Performance Bounds for Linear MMSE Receivers in CDMA Systems with Partial Channel Knowledge

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Abstract

In CDMA systems employing linear adaptive receivers, the detector is typically estimated directly from the received signals, based on some partial knowledge about the system, e.g., signature waveforms of one or several users. We derive the Cramer-Rao lower bounds on the covariances of the estimated linear detectors, under three different assumptions on the mechanism for estimating the detectors, namely, (a) finite-alphabet-based (FA) blind detectors; (b) constant-modulus-based (CM) blind detectors; and (c) second-order-moments-based (SO) blind detectors. These bounds translate into the upper bounds on the achievable SINR by the corresponding adaptive receivers. The results are asymptotic in nature, either for high signal-to-noise ratio or for large signal sample size. The effects of unknown multipath channels on these performance bounds are also addressed. Numerical results indicate that while the existing subspace blind or group-blind detectors perform close to the SINR bound for the SO detectors, the SINR bounds for the FA and CM detectors are significantly higher, which suggests potential avenues for developing more powerful adaptive detectors by exploiting more structural information of the system.

Key Words: CDMA, adaptive linear MMSE detector, Cramer-Rao lower bound, multipath channels, finite-alphabet, constant-modulus, second-order moments.

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1 Introduction

Adaptive multiuser detection is a subject of intensive research in recent years [2, 3]. Of particular interest is the so-called blind and group-blind linear multiuser detection [1, 20, 22, 23]. The basic scenario is that the receiver has only limited knowledge about the CDMA channel, e.g., the signature waveform of a given user (or the signature waveforms of some but not all users), based on which a linear detector is estimated from the received signals and subsequently employed to demodulate the data symbols of the given user.

In this paper, we address the following fundamental problem: Given a particular linear detector, e.g., the linear MMSE detector, and given certain prior knowledge about the channel, what is the best achievable performance of the corresponding adaptive implementation of this linear detector, in terms of the output signal-to-interference-plus-noise ratio (SINR)? To be more specific, consider a synchronous CDMA system with $K$ users, employing signature waveforms $s_1, \ldots, s_K$, (Here the signature waveforms are defined at the receiver, and therefore they are obtained as a convolution of the CDMA code and the channel, which implies that different received signals may have different powers.) with processing gain $N$, and transmitting data symbols $b_1[m], \ldots, b_K[m]$ at time $m$. The received signal can be written as

$$r[m] = \sum_{k=1}^{K} b_k[m] s_k + n[m] = S b[m] + n[m], \quad m = 1, \ldots, M,$$

where $S \triangleq [s_1 \ s_2 \ \ldots \ s_K]$, $b[m] \triangleq [b_1[m] \ b_2[m] \ \ldots \ b_K[m]]^T$, and $n[m]$ is the ambient Gaussian noise. In linear detection for a given user, say user 1, a linear detector $w_1$ is correlated with the received signal to obtain the decision statistic $w_1^T r[m]$, based on which a decision on symbol $b_1[m]$ is made. In general, $w_1$ is a function of the signature waveforms of all users. In blind and group-blind multiuser detection, only an estimate $\hat{w}_1$ of $w_1$ can be obtained from a set of received signals $\{r[m]\}$ of $M$ symbols, and based on some partial knowledge about the channel.

The problem of estimating the linear detector $w_1$ from the received signals is essentially a parameter estimation problem. The Cramer-Rao bound (CRB) gives a lower bound on the covariance of the estimation error $(\hat{w}_1 - w_1)$, which as will be shown in Section 5, translates into an upper bound on the achievable SINR by the estimated detector. Note that different assumptions on the prior knowledge about the system that the estimator can make use of result in different CRB’s.
(and therefore different upper bounds on the achievable SINR). In particular, the more structural information the detector can exploit, the better is the potential performance of this detector; and at the same time, usually the more complex it is to implement such a detector. In this paper, we consider the following three scenarios for estimating $\mathbf{w}_1$, with the common assumption being that the signature waveforms of the first $\bar{K}$ users, $\mathbf{s}_1, \ldots, \mathbf{s}_{\bar{K}}$, are known to the receiver; in addition,

- Finite-alphabet-based (FA) estimates: the user symbol constellation (e.g., BPSK or QPSK) is assumed known; and a set of received signals $\{\mathbf{r}[m]\}_{m=1}^{M}$ is available.

- Constant-modulus-based (CM) estimates: the user symbols are assumed to have constant modulus; and a set of received signals $\{\mathbf{r}[m]\}_{m=1}^{M}$ is available.

- Second-order-moments-based (SO) estimates: only the sample autocorrelation $\hat{C}_r = \sum_{m=1}^{M} \mathbf{r}[m] \mathbf{r}[m]^T$ of the received signals is available.

Under each of the above three assumptions, we derive the CRB for $\hat{\mathbf{w}}_1$ and the corresponding upper bound on the achievable SINR. Note that it is assumed that no structural information on the remaining $(K - \bar{K})$ users’ signature waveforms is exploited by the receiver. The bounds for the SO case are asymptotic in $M$. The bounds for the FA and CM cases are valid for any values of $M$ and SNR, but they are computationally prohibitive to evaluate. We therefore derive asymptotic bounds that can easily be calculated: for high SNR in the FA case and for large $M$ in the CM case.

We also derive the performance bounds for blind detectors in systems with unknown multipath channels. We remark that the problems considered here are related to the blind source separation problem [6, 15, 16], although in the context of linear adaptive detection in CDMA systems, the linear detector $\hat{\mathbf{w}}_1$ is generally uniquely defined, whereas in blind source separation there is usually a problem of uniqueness.

## 2 CRB for FA-Estimates of Linear MMSE Detector

In this section, it is assumed that the linear MMSE detector is estimated based on the knowledge of the symbol constellation (e.g., BPSK or QPSK) and a set of received signals $\{\mathbf{r}[m]\}_{m=1}^{M}$. 

2
2.1 Real-valued Signals

Consider the signal model (1), where \( s_k \in \mathbb{R}^N \) and \( b_k[m] \in \{+1, -1\} \) are respectively the received signature waveform and the \( m \)-th transmitted data symbol of the \( k \)-th user; \( n[m] \sim \mathcal{N}(0, \eta I_N) \). It is assumed that the user data symbols \( \{b_k[m]\} \) are i.i.d. and equiprobable; and the noise vectors \( \{n[m]\} \) are also i.i.d. The linear MMSE detector for user 1 is given by

\[
\mathbf{w}_1 = C_r^{-1} s_1 = (S S^T + \eta I_N)^{-1} s_1, \quad \text{with} \quad C_r \triangleq E \{ r[m] r[m]^T \} = S S^T + \eta I_N
\]  

(2)

being the autocorrelation matrix of the received signals.

It is assumed that the signature waveforms of the first \( K \) users, \( s_1, s_2, \ldots, s_K \), are known to the receiver; whereas those of the remaining \( \tilde{K} \) users are unknown. For simplicity, assume that the noise variance \( \eta \) is also known to the receiver. In this section, it is also assumed that the receiver knows the modulation format of the data symbols (i.e., BPSK) of all users. Denote \( \hat{\mathbf{S}} \triangleq [s_1 \ s_2 \ldots \ s_K], \ S \triangleq [s_{K+1} \ldots \ s_K], \ \hat{\mathbf{b}} \triangleq [b_1 \ b_2 \ldots \ b_K]^T, \) and \( \mathbf{b} \triangleq [b_{K+1} \ldots \ b_K]^T. \) Denote \( \phi_\eta(x) \triangleq (2\pi\eta)^{-\frac{N}{2}} \exp \left(-\frac{x^2}{2\eta} \right). \) Then the pdf of the received signal at any time given the unknown parameters \( \tilde{\mathbf{S}} \) can be written as (Here for simplicity, we drop the time index \( m. \))

\[
f(r; \tilde{\mathbf{S}}) = 2^{-K} \sum_{\mathbf{b} \in \{+1, -1\}^K} \phi_\eta \left( \|r - \hat{\mathbf{S}} \mathbf{b} - \mathbf{S} \mathbf{b}\| \right).
\]

(3)

Denote by \( \text{vec}(\tilde{\mathbf{S}}) \) the vectorization of the matrix \( \tilde{\mathbf{S}} \), i.e., \( \text{vec}(\tilde{\mathbf{S}})_{m+nN} = [\tilde{\mathbf{S}}]_{m,n} \). The Fisher information matrix [11, 17] for \( \tilde{\mathbf{S}} \) is an \( NK \times NK \) matrix, given by

\[
J_{\tilde{\mathbf{S}}} = E \left\{ \left[ \frac{\partial}{\partial \text{vec}(\mathbf{S})} \ln f(r; \tilde{\mathbf{S}}) \right] \left[ \frac{\partial}{\partial \text{vec}(\mathbf{S})} \ln f(r; \tilde{\mathbf{S}}) \right]^T \right\}.
\]

(4)

Let \( \mathbf{w}_1 \) be an unbiased estimate of the linear MMSE detector \( \mathbf{w}_1 \) based on \( M \) i.i.d. received signals \( \{r[m]\}_{m=1}^M \). Then the following Cramer-Rao lower bound on the covariance matrix of the estimated detector holds [17]

\[
E \left\{ \left( \mathbf{w}_1 - \mathbf{w} \right) \left( \mathbf{w}_1 - \mathbf{w} \right)^T \right\} \geq K_w \triangleq \frac{1}{M} H^T J^{-1} S H,
\]

(5)

where \( H \) is an \( NK \times N \) matrix with elements given by \( H_{(m,n),k} = \frac{\partial n_k}{\partial s_{m,n}} \), with \( n_k \triangleq [n[k]] \) and \( \tilde{S}_{m,n} \triangleq [\tilde{\mathbf{S}}]_{m,n}; \) and where the notation \( A \geq B \) means that the matrix \( A - B \) is positive semidefinite. Note that here we use the double index \( (m,n) \) in \( H_{(m,n),k} \) to denote row \( m + nN \). The
matrix $H$ is most easily found using the matrix differential calculus [4, 14]. Since (2) can be rewritten as $w_1 = \left( \tilde{S}\tilde{S}^T + \bar{S}\bar{S}^T + \eta I_N \right)^{-1} s_1$, we have the following differential

$$
\Delta w_1 = -C_r^{-1} \left( \tilde{S} \Delta \tilde{S}^T + \Delta \tilde{S} \tilde{S}^T \right) C_r^{-1} s_1,
$$

$$\implies H_{(m,n),k} = -[C_r^{-1} \tilde{S}]_{k,n} [C_r^{-1} s_1]_m - [C_r^{-1}]_{k,m} [\tilde{S}^T C_r^{-1} s_1]_n. \tag{7}
$$

The elements of the Fisher information matrix $J_{\tilde{S}}$ in (4) can be calculated as follows:

$$
J_{(i,j)(k,l)} = E \left\{ \frac{1}{f(r; \tilde{S})^2} \left( \frac{\partial f(r; \tilde{S})}{\partial S_{i,j}} \right) \left( \frac{\partial f(r; \tilde{S})}{\partial S_{k,l}} \right) \right\}^{-1} - \sum_b \tilde{b}_j \left( r_i - \xi_i^T b \right) \phi_\eta (\|r - Sb\|) \sum_b \tilde{b}_l \left( r_k - \xi_k^T b \right) \phi_\eta (\|r - Sb\|) \, dr, \tag{8}
$$

where $\tilde{b}_j \triangleq [b]_j$, $r_i \triangleq [r]_i$, and $\xi_i^T$ denotes the $i$-th row of $S$. Hence it is prohibitive to evaluate $J_{\tilde{S}}$. We instead consider its asymptotic expression at the high SNR region. We have the following result. The proof is given in Appendix A.

**Proposition 1** The information matrix $J_{\tilde{S}}$ given by (4) has the following limit

$$
\lim_{\eta \to 0} \eta J_{(i,j)(k,l)} = \delta_{i,k} \delta_{j,l}, \quad \text{or} \quad \lim_{\eta \to 0} \eta J_{\tilde{S}} = I_{NN}. \tag{9}
$$

Substituting (9) into (5), the asymptotic CRB (for high SNR) on $\hat{w}_1$ is then given by

$$
K_w \approx \frac{\eta}{M} H^T H. \tag{10}
$$

Using (7), after some manipulations, we obtain

$$
[H^T H]_{k,l} = \sum_{i,j} H_{(i,j),k} H_{(i,j),l} = \left( s_i^T C_r^{-2} s_1 \right) \left[ C_r^{-1} \tilde{S}\tilde{S}^T C_r^{-1} s_1 \right]_{k,l} + \left( s_i^T C_r^{-1} \tilde{S}\tilde{S}^T C_r^{-1} s_1 \right) \left[ C_r^{-2} \right]_{k,l} + \left[ C_r^{-1} \tilde{S}\tilde{S}^T C_r^{-1} s_1 s_1^T C_r^{-2} \right]_{k,l} + \left[ C_r^{-2} s_1 s_1^T C_r^{-1} \tilde{S}\tilde{S}^T C_r^{-1} \right]_{k,l}. \tag{11}
$$

Substituting (11) into (10), we obtain the following asymptotic CRB on $\hat{w}_1$ at high SNR:

$$
K_w \approx \frac{\eta}{M} \left( C_r^{-1} \tilde{S}\tilde{S}^T C_r^{-1} s_1 s_1^T C_r^{-2} s_1 + C_r^{-2} s_1^T C_r^{-1} \tilde{S}\tilde{S}^T C_r^{-1} s_1 + C_r^{-1} \tilde{S}\tilde{S}^T C_r^{-1} s_1 s_1^T C_r^{-2} + C_r^{-2} s_1 s_1^T C_r^{-1} \tilde{S}\tilde{S}^T C_r^{-1} \right). \tag{12}
$$
2.2 Complex-valued Signals

Now assume that the signals in (1) are complex-valued. In particular, \( \{b_k[m]\} \) are i.i.d. equiprobable BPSK or QPSK symbols, \( s_k \in \mathbb{C}^N \) is the complex composite signature waveform of the \( k \)-th user, and \( n[m] \sim \mathcal{N}(0, \eta I_N) \). The autocorrelation matrix of the received signals becomes \( C_r = E \{r[m]r[m]^H\} = SS^H + \eta I_N \); and the linear MMSE detector for user 1 is given by

\[
  w_1 = C_r^{-1}s_1 = (SS^H + \eta I_N)^{-1}s_1. \tag{13}
\]

Denote \( \psi_\eta(x) = (\pi \eta)^{-N} \exp \left(-\frac{x^2}{\eta}\right) \). The pdf of the received signal given the unknown parameters \( \bar{S} \) is

\[
f(r; \bar{S}) = 2^{-qK} \sum_b \psi_\eta \left( ||r - \bar{S}b - \bar{S}b|| \right), \tag{14}
\]

where \( q = 1 \) for BPSK and \( q = 2 \) for QPSK. The Fisher information matrix \( J_{\bar{s}} \) for \( [\Re \bar{S} \Im \bar{S}] \) is a \( 2NK \times 2NK \) matrix, with

\[
  J_{x,y}^{(i,j)(k,l)} = E \left\{ \frac{1}{f(r; \bar{S})^2} \frac{\partial f(r; \bar{S})}{\partial S^{(x)}_{i,j}} \frac{\partial f(r; \bar{S})}{\partial S^{(y)}_{k,l}} \right\}, \tag{15}
\]

where \( x, y \in \{r, i\} \) indicates real respectively imaginary part, and \( S^{(r)}_{i,j} = \Re S_{i,j}, S^{(i)}_{i,j} = \Im S_{i,j} \). The derivatives are given by

\[
  \frac{\partial f(r; \bar{S})}{\partial S^{(r)}_{i,j}} = 2^{-qK} \eta \sum_b \left[ \Re (r_i - \xi^{(r)}_{i,b}) \Re \bar{b}_j + \Im (r_i - \xi^{(r)}_{i,b}) \Im \bar{b}_j \right] \psi_\eta \left( ||r - \bar{S}b|| \right), \tag{16}
\]

\[
  \frac{\partial f(r; \bar{S})}{\partial S^{(i)}_{i,j}} = 2^{-qK} \eta \sum_b \left[ \Im (r_i - \xi^{(i)}_{i,b}) \Im \bar{b}_j - \Re (r_i - \xi^{(i)}_{i,b}) \Re \bar{b}_j \right] \psi_\eta \left( ||r - \bar{S}b|| \right), \tag{17}
\]

where \( \xi^{(i)}_{i,b} \) denotes the \( i \)-th row of \( S \). Note that (16) and (17) can be rewritten as follows:

\[
  \frac{\partial f(r; \bar{S})}{\partial S^{(x)}_{i,j}} = 2^{-qK} \eta \sum_b \left\{ (r_i - \xi^{(i)}_{i,b})^{(x)} \Re \bar{b}_j + [-J (r_i - \xi^{(i)}_{i,b})]^{(x)} \Im \bar{b}_j \right\} \psi_\eta \left( ||r - \bar{S}b|| \right), \tag{18}
\]

with \( x \in \{r, i\} \). We have the following asymptotic expression for \( J_{\bar{s}} \) at high SNR. The proof is given in Appendix A.

**Proposition 2** The Fisher information matrix \( J_{\bar{s}} \) given by (15) has the following limit

\[
  \lim_{\eta \to 0} \eta J_{x,y}^{(i,j)(k,l)} = 2 \delta_{i,k} \delta_{j,l} \delta_{x,y}, \quad \text{or} \quad \lim_{\eta \to 0} \eta J_{\bar{s}} = 2I_{2NK}. \tag{19}
\]
For any estimate \( \hat{w}_1 \in \mathbb{C}^N \) of the linear MMSE detector \( w_1 \), its covariance is characterized by the following 2\( N \times 2\) real-valued matrix,

\[
\text{Cov} \begin{bmatrix} \Re \hat{w}_1 \\ \Im \hat{w}_1 \end{bmatrix} = \begin{bmatrix} \text{Cov} \{ \Re \hat{w}_1, \Re \hat{w}_1 \} & \text{Cov} \{ \Re \hat{w}_1, \Im \hat{w}_1 \} \\ \text{Cov} \{ \Im \hat{w}_1, \Re \hat{w}_1 \} & \text{Cov} \{ \Im \hat{w}_1, \Im \hat{w}_1 \} \end{bmatrix}.
\]

(20)

An equivalent characterization is through the following two complex-valued covariance matrices

\[
C_w \triangleq E \{ (\hat{w}_1 - w_1)(\hat{w}_1 - w_1)^H \}, \quad \text{and} \quad \hat{C}_w \triangleq E \{ (\hat{w}_1 - w_1)(\hat{w}_1 - w_1)^T \},
\]

with the following equivalence relationship

\[
\text{Cov} \begin{bmatrix} \Re \hat{w}_1 \\ \Im \hat{w}_1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \Re C_w + \Re \hat{C}_w & \Im C_w - \Im \hat{C}_w \\ \Im C_w + \Im \hat{C}_w & \Re C_w - \Re \hat{C}_w \end{bmatrix},
\]

(21)

or

\[
C_w = \text{Cov} \{ \Re \hat{w}_1, \Re \hat{w}_1 \} + \text{Cov} \{ \Im \hat{w}_1, \Im \hat{w}_1 \} + j \left( \text{Cov} \{ \Im \hat{w}_1, \Re \hat{w}_1 \} - \text{Cov} \{ \Re \hat{w}_1, \Im \hat{w}_1 \} \right),
\]

(22)

\[
\hat{C}_w = \text{Cov} \{ \Re \hat{w}_1, \Re \hat{w}_1 \} - \text{Cov} \{ \Im \hat{w}_1, \Im \hat{w}_1 \} + j \left( \text{Cov} \{ \Im \hat{w}_1, \Re \hat{w}_1 \} + \text{Cov} \{ \Re \hat{w}_1, \Im \hat{w}_1 \} \right).
\]

(23)

Let \( \hat{w}_1 \) be an unbiased estimate of the linear MMSE detector \( w_1 \) based on \( M \) i.i.d. received signals \( \{ r[m] \}_{m=1}^M \). Then the Cramer-Rao lower bound can be written as

\[
\text{Cov} \begin{bmatrix} \Re \hat{w}_1 \\ \Im \hat{w}_1 \end{bmatrix} \preceq \begin{bmatrix} K_w^{rr} & K_w^{ri} \\ K_w^{ir} & K_w^{ii} \end{bmatrix} \preceq \frac{1}{M} \mathcal{H}^T J_s^\dagger J_s \mathcal{H},
\]

(24)

where \( \mathcal{H} \) is a \( 2N \times 2N \) matrix whose elements are given by \( \mathcal{H}_{(i,j),k}^{x,y} = \frac{\partial u_k^{(y)}}{\partial \bar{s}_{k,j}} \), where \( x, y \in \{ r, i \} \). It can be shown that [25]

\[
\mathcal{H} = \begin{bmatrix} \Re \mathcal{H} + \Re \bar{\mathcal{H}} & \Im \mathcal{H} + \Im \bar{\mathcal{H}} \\ \Im \mathcal{H} - \Im \bar{\mathcal{H}} & \Re \mathcal{H} - \Re \bar{\mathcal{H}} \end{bmatrix},
\]

(25)

where \( \mathcal{H} \) and \( \bar{\mathcal{H}} \) are \( N \times N \) matrices whose elements are given respectively by \( H_{(i,j),k} = \frac{\partial u_k}{\partial \bar{s}_{k,j}} \), and \( \bar{H}_{(i,j),k} = \frac{\partial u_k}{\partial s_{k,j}} \). Since \( w_1 = C_r^{-1} s_1 = \left( \bar{S} \bar{S}^H + \bar{S} \bar{S}^H + \eta I_N \right)^{-1} s_1 \), taking the differential we have \( \Delta w_1 = -C_r^{-1} \left( \bar{S} \Delta \bar{S}^H + \Delta \bar{S} \bar{S}^H \right) C_r^{-1} s_1 \). Therefore we obtain

\[
H_{(i,j),k} = \left[ C_r^{-1} s_1 \right]_{[i,j]} \left( \bar{S} \Delta \bar{S}^H \right) \left[ C_r^{-1} s_1 \right]_{[i,j]} \quad \text{and} \quad \bar{H}_{(i,j),k} = \left[ C_r^{-1} \bar{S} \bar{S} \right]_{[k,j]} \left[ C_r^{-1} s_1 \right]_{[i,j]}.
\]

(26)

Now similar to (22) and (23), define

\[
K_w = K_w^{rr} + K_w^{ri} + j \left( K_w^{ir} - K_w^{ii} \right), \quad \text{and} \quad \bar{K}_w = K_w^{rr} - K_w^{ii} + j \left( K_w^{ir} + K_w^{ii} \right).
\]

(27)
Using (24), (25) and $J^\dagger_s \cong \frac{\eta}{2} I_{2NK}$, after some tedious but straightforward algebra, (27) can be written in terms of $H$ and $\hat{H}$ as (at high SNR)

$$K_w \cong \frac{\eta}{M} \left( H^H H + \hat{H}^H \hat{H} \right)^\dagger, \quad \text{and} \quad \tilde{K}_w \cong \frac{\eta}{M} \left( H^T \tilde{H} + \hat{H}^T \hat{H} \right).$$

(28)

Using (26), after some manipulations, we obtain $[H^H H]_{k,l} = [C_r^{-2}]_{k,l} (s_1^H C_r^{-1} \bar{S} S^H C_r^{-1} s_1)$, $[\hat{H}^H \hat{H}]_{k,l} = [C_r^{-1} \bar{S}^H S C_r^{-1}]_{k,l} (s_1^H C_r^{-2} s_1)$, and $[H^T \tilde{H}]_{k,l} = [C_r^{-2} s_1]_{k} [C_r^{-1} \bar{S} S^H C_r^{-1} s_1]$. Substituting these into (28) we obtain the following asymptotic CRB on $\hat{w}$ at high SNR

$$K_w \cong \frac{\eta}{M} \left[ (s_1^H C_r^{-1} \bar{S} S^H C_r^{-1} s_1) C_r^{-2} + (s_1^H C_r^{-2} s_1) C_r^{-1} \bar{S} S^H C_r^{-1} \right],$$

(29)

and

$$\tilde{K}_w \cong \frac{\eta}{M} \left[ C_r^{-2} s_1 s_1^T C_r^{-T} \bar{S}^* S^T C_r^{-T} + C_r^{-1} \bar{S} S^H C_r^{-1} s_1 s_1^T (C_r^{-2})^T \right].$$

(30)

\section{CRB for SO-Estimates of Linear MMSE Detector}

In this section, it is assumed that the linear MMSE detector is estimated based on only the sample autocorrelation $\hat{C}_r = \sum_{m=1}^M r[m] r[m]^T$ of the received signals.

\subsection{Real-valued Signals}

We will make use of the following result, which is an extension of Theorem 2 of [12]. The proof is given in Appendix B.

**Proposition 3** Let $\{r[m]\}_{m=1}^M$ be a set of i.i.d. observations whose pdf is parameterized by $\theta$. Let $\hat{x}(M)$ be a statistic of the observations (with dimension independent of $M$), and suppose that

$$\lim_{M \to \infty} \hat{x}(M) = x = f(\theta) \quad \text{almost surely, where } f \text{ is a } C^2 \text{ function. Let } \alpha = g(\theta), \text{ and suppose there exists some function } h \text{ so that } g(\theta) = h(f(\theta)). \text{ Let } \hat{\alpha}(\hat{x}(M)) \text{ be a consistent estimate of } \alpha \text{ based solely on } \hat{x}(M). \text{ Assume that the technical conditions in [12] hold. Then}

$$\lim_{M \to \infty} M \cdot E \left\{ (\hat{\alpha} - \alpha) (\hat{\alpha} - \alpha)^T \right\} \geq \frac{\partial g}{\partial \theta} \left( \frac{\partial f}{\partial \theta} \right)^T \Sigma^{-1} \frac{\partial f}{\partial \theta},$$

(31)

where $\Sigma = \lim_{M \to \infty} M \cdot \text{Cov} \left\{ \hat{x}(M) \right\}$.

Note that this has the same form as the Gaussian CRB [17] and can be considered as an asymptotic CRB, where $\left( \frac{\partial f}{\partial \theta} \right)^T \Sigma^{-1} \frac{\partial f}{\partial \theta}$ plays the role of the Fisher information matrix.
Consider the real-valued signal model (1) and assume BPSK modulation. In this section we consider estimates $\hat{w}_1$ of the linear detector $w_1$ that are functions solely of the following sample autocorrelation matrix $\hat{C}_r(M) \triangleq \frac{1}{M} \sum_{m=1}^{M} r[m] r[m]^T$. We aim to find the bounds for such estimates. Since both $C_r$ and $\hat{C}_r(M)$ are symmetric, they contain only $\frac{1}{2}(N^2 + N)$ distinct elements. Define the statistic $\hat{x}(M)$ as the vector containing the distinct elements of $\hat{C}_r(M)$ and $x$ as the corresponding elements of $C_r$, e.g.,

$$[x]_{(i-1)/2+j} = C_{i,j}, \quad \text{and} \quad [\hat{x}(M)]_{(i-1)/2+j} = \hat{C}_{i,j}(M), \quad i = 1, \ldots, N; \quad j = 1, \ldots, i. \quad (32)$$

Define the matrix $J_S$ by

$$J_{(i,j),(k,l)} = \frac{\partial x^T}{\partial S_{i,j}} \Sigma^{-1} \frac{\partial x}{\partial S_{k,l}}, \quad (33)$$

where the $\frac{1}{2}(N^2 + N) \times \frac{1}{2}(N^2 + N)$ matrix $\Sigma \triangleq \lim_{M \to \infty} MCov\{\hat{x}(M)\}$. This covariance matrix is found in [4] as

$$\Sigma_{(i,j)(k,l)} = C_{i,k}C_{j,l} + C_{i,l}C_{j,k} - 2 \sum_{\alpha=1}^{K} S_{i,\alpha} S_{j,\alpha} S_{k,\alpha} S_{l,\alpha}. \quad (34)$$

Since $C_r = SS^T + \tilde{S} \tilde{S}^T + \eta I_N$, we have

$$\left[ \frac{\partial x}{\partial S_{i,j}} \right]_{(m-1)/2+n} = \frac{\partial C_{m,n}}{\partial S_{i,j}} = \delta_{m,n} \delta_{i,j} + \delta_{m,i} \delta_{n,j} + \delta_{m,j} \delta_{n,i}. \quad (35)$$

Let $\hat{w}_1$ be a consistent estimate of the linear MMSE detector $w_1$ from the sample autocorrelation matrix $\hat{C}_r(M)$, without knowing the signature waveforms $S$. According to Proposition 3 we then have the following asymptotic lower bound, where the matrix $H$ is given by (7):

$$\lim_{M \to \infty} ME\left\{ (\hat{w}_1 - w_1) (\hat{w}_1 - w_1)^T \right\} \geq H^T J_S^T H. \quad (36)$$

Hence for large $M$, we have $K_w \cong \frac{1}{M} H^T J_S^T H$.

### 3.2 Complex-valued Signals

Now assume that the signals in (1) are complex-valued and either BPSK or QPSK modulation is employed. Suppose that an estimate $\hat{w}_1$ of the linear MMSE detector $w_1$ is based on the following sample autocorrelation matrix $\hat{C}_r(M) \triangleq \frac{1}{M} \sum_{m=1}^{M} r[m] r[m]^H$. Since both $C_r$ and $\hat{C}_r(M)$ are
Hermitean, they contain $\frac{1}{2}(N^2 - N)$ distinct complex (off-diagonal) elements, and $N$ real (diagonal) elements. Let $x$ and $\hat{x}(M)$ be the $N^2$-dimensional real vector containing the real and imaginary parts of the distinct elements of $C_r$ and $\hat{C}_r(M)$ respectively. Define the (real) matrix $J_S$ as

$$ J^{x,y}_{(i,j),(k,l)} \triangleq \frac{\partial x^T}{\partial S_{i,j}^{(x)}} \Sigma^{-1} \frac{\partial x}{\partial S_{k,l}^{(y)}}, \quad (37) $$

where $x, y \in \{r, i\}$ indicates real or imaginary part, and where the $N^2 \times N^2$ matrix $\Sigma \triangleq \lim_{M \to \infty} M \text{Cov}\{\hat{x}(M)\}$ is specified by

$$ \Sigma^{x,y}_{(i,j),(k,l)} = \lim_{M \to \infty} M \cdot \text{Cov}\left\{\hat{C}_{i,j}^{(x)}, \hat{C}_{k,l}^{(y)}\right\} = \begin{cases} \frac{1}{2} (\Re \Gamma_{(i,j),(k,l)} - \Re \Gamma_{(i,j),(k,l)}), & \text{if } x = r, y = r \\ \frac{1}{2} (\Im \Gamma_{(i,j),(k,l)} - \Im \Gamma_{(i,j),(k,l)}), & \text{if } x = r, y = i \\ \frac{1}{2} (\Re \Gamma_{(i,j),(k,l)} - \Re \Gamma_{(i,j),(k,l)}), & \text{if } x = i, y = r \\ \frac{1}{2} (\Im \Gamma_{(i,j),(k,l)} - \Im \Gamma_{(i,j),(k,l)}), & \text{if } x = i, y = i \end{cases}. \quad (38) $$

In (38), the expressions for $\Gamma_{(i,j),(k,l)}$ and $\hat{\Gamma}_{(i,j),(k,l)}$ are given by [4]

$$ \Gamma_{(i,j),(k,l)} \triangleq \lim_{M \to \infty} M \text{Cov}\left\{\hat{C}_{i,j}^{r}, \hat{C}_{k,l}^{r}\right\} = C_{i,k}^{r} C_{j,l}^{r} + \mu [SS^T]_{i,l}[SS^T]_{j,k} + \nu \sum_{\alpha=1}^{K} \delta_{i,\alpha} S_{j,\alpha}^{*} S_{k,\alpha}^{*} S_{l,\alpha}, \quad (39) $$

$$ \hat{\Gamma}_{(i,j),(k,l)} \triangleq \lim_{M \to \infty} M \text{Cov}\left\{\hat{C}_{i,j}^{r}, \hat{C}_{k,l}^{r}\right\} = C_{i,k}^{r} C_{j,l}^{r} + \mu [SS^T]_{i,k}[SS^T]_{j,l} + \nu \sum_{\alpha=1}^{K} \delta_{i,\alpha} S_{j,\alpha}^{*} S_{k,\alpha}^{*} S_{l,\alpha}, \quad (40) $$

with $\mu = 1$, $\nu = -2$ for BPSK; and $\mu = 0$, $\nu = -1$ for QPSK.

Moreover, since $C_r = SS^H + SS^H + \eta I_N$, we have

$$ \frac{\partial C_{m,n}}{\partial S_{i,j}} = \delta_{m,i} S_{n,j}^{*}, \quad \text{and} \quad \frac{\partial C_{m,n}}{\partial S_{i,j}^{*}} = \delta_{n,j} S_{m,i}. \quad (41) $$

Hence

$$ \frac{\partial C_{m,n}}{\partial S_{i,j}^{(x)}} = \begin{cases} \Re \left(\delta_{m,i} \bar{S}_{n,j}^{(x)}\right) + \Re \left(\delta_{n,j} \bar{S}_{m,i}^{(x)}\right), & \text{if } x = r, y = r \\ \Im \left(\delta_{m,i} \bar{S}_{n,j}^{(x)}\right) + \Im \left(\delta_{n,j} \bar{S}_{m,i}^{(x)}\right), & \text{if } x = r, y = i \\ \Re \left(\delta_{m,i} \bar{S}_{n,j}^{(x)}\right) - \Re \left(\delta_{n,j} \bar{S}_{m,i}^{(x)}\right), & \text{if } x = i, y = r \\ \Im \left(\delta_{m,i} \bar{S}_{n,j}^{(x)}\right) - \Im \left(\delta_{n,j} \bar{S}_{m,i}^{(x)}\right), & \text{if } x = i, y = i \end{cases}, \quad (42) $$

based on (41) and (42) $\partial x / \partial S_{i,j}^{(y)}$ in (37) can be obtained.

The bound from Proposition 3 is now given by

$$ \lim_{M \to \infty} M \text{Cov}\left\{\begin{bmatrix} \Re w_1 \\ \Im w_1 \end{bmatrix}\right\} \geq \mathcal{H}^T J_S^\dagger \mathcal{H} \implies \begin{bmatrix} K_w^{rr} & K_w^{ri} \\ K_w^{ir} & K_w^{ii} \end{bmatrix} \cong \frac{1}{M} \mathcal{H}^T J_S^\dagger \mathcal{H}, \quad (43) $$
where $J_S$ is given by (37)-(42), and the transformation matrix $H$ is given by (25) and (26). Finally $K_w$ and $K_w$ are given by (27).

4 CRB for CM-Estimates of Linear MMSE Detector

In this section, it is assumed that the linear MMSE detector is estimated based on the knowledge that the data symbols have the constant modulus property, and a set of received signals $\{r[m]\}_{m=1}^M$. Specifically, in such a scenario, in addition to the unknown signature waveforms $S$, the transmitted data symbols $b[m] \in \mathbb{C}^K$ are also considered as unknown parameters. Moreover, we constrain these symbols to have the constant modulus property, i.e., $b_k[m] = \exp(\phi_k[m]), -\pi \leq \phi_k[m] < \pi$.

To resolve the possible phase ambiguity between $s_k$ and $b_k$, we assume that $b[1]$ is known. Denote $\tilde{\phi}[1] \triangleq [\phi_1[1], \ldots, \phi_K[1]]^T$ and $\phi[m] \triangleq [\phi_1[m], \ldots, \phi_K[m]]^T, m = 2, \ldots, M$. Then the unknown parameters in this system are

$$\omega \triangleq [\tilde{\phi}[1]^T \phi[2]^T \ldots \phi[M]^T \text{vec}(\Re S)^T \text{vec}(\Im S)^T]^T.$$ (44)

Denote $B[m] \triangleq b[m]^T \otimes I_N$. The complex version of the received signal (1) can be rewritten as

$$r[m] = S \exp(j\phi[m]) + n[m]$$ (45)

$$= \tilde{S} \bar{b}[m] + [B[m] \ jB[m]] \begin{bmatrix} \text{vec}(\Re S) \\ \text{vec}(\Im S) \end{bmatrix} + n[m].$$ (46)

Denote $Y \triangleq \text{vec}[r[1], \ldots r[M]]$, and $\mu \triangleq \text{vec}[Sb[1], \ldots, S\bar{b}[M]]$. Since $n[m] \overset{\text{i.i.d.}}{\sim} \mathcal{CN}(0, \eta I_N)$, the Fisher information matrix of $\omega$ based on $Y$ is given by [7]

$$J_\omega = E \left\{ \frac{\partial}{\partial \omega} \ln f(Y; \omega) \left[ \frac{\partial}{\partial \omega} \ln f(Y; \omega) \right]^T \right\} = 2 \Re \left\{ \frac{\partial \mu^H}{\partial \omega} \frac{\partial \mu}{\partial \omega} \right\}. \hspace{1cm} (47)$$

Define $\bar{\beta}[1] \triangleq \text{diag}(\bar{b}[1])$ and $\beta[m] \triangleq \text{diag}(b[m])$. Using (45) and (46), we have

$$\frac{\partial \mu}{\partial \omega} = \begin{bmatrix} j\bar{S} \bar{\beta}[1] & 0 & \cdots & 0 & \bar{B}[1] & j\bar{B}[1] \\ 0 & jS \bar{\beta}[2] & \cdots & 0 & \bar{B}[2] & j\bar{B}[2] \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & jS \beta[M] & \bar{B}[M] & j\bar{B}[M] \end{bmatrix}. \hspace{1cm} (48)$$
Substituting (48) into (47) we obtain

\[ J_\omega = \frac{2}{\eta} \begin{bmatrix} J_{11} & J_{12} \\ J_{12}^T & J_{22} \end{bmatrix}, \]  

(49)

where

\[ J_{11} = \text{diag}(H[1], \ldots, H[M]), \]  

(50)

\[ H[1] = \Re \left( \bar{\beta}[1] H_2 \bar{S} \beta[1] \right), \]  

(51)

\[ H[m] = \Re \left( \beta[m] H_2 S \bar{S} \beta[m] \right), \quad m = 2, \ldots, M; \]  

(52)

\[ J_{22} = \sum_{m=1}^{M} \Re \left\{ [B[m] \bar{B}[m]] H [B[m] \bar{B}[m]] \right\} \]

\[ = \sum_{m=1}^{M} \begin{bmatrix} \Re \{ B[m] H \bar{B}[m] \} & -\Im \{ B[m] H \bar{B}[m] \} \\ \Im \{ B[m] H \bar{B}[m] \} & \Re \{ B[m] H \bar{B}[m] \} \end{bmatrix} \]  

(53)

\[ J_{22}[m] \]

and


(54)

\[ G[1] = \Re \left\{ -J \bar{\beta}[1] H_2 \bar{S} \beta[1], \bar{B}[1] \right\} \]

\[ = \begin{bmatrix} \Im \left\{ \bar{\beta}[1] H_2 \bar{S} \beta[1] \right\} , \Re \left\{ \bar{\beta}[1] H_2 \bar{S} \beta[1] \right\} \end{bmatrix}, \]  

(55)

\[ G[m] = \Re \left\{ -J \beta[m] H_2 S \bar{S} \beta[m], \bar{B}[m] \right\} \]

\[ = \begin{bmatrix} \Im \left\{ \beta[m] H_2 S \bar{S} \beta[m] \right\} , \Re \left\{ \beta[m] H_2 S \bar{S} \beta[m] \right\} \end{bmatrix}. \]  

(56)

The above Fisher information matrix is computationally difficult to evaluate because it has to be averaged over all possible transmitted symbols. We have the following result regarding the asymptotic invertibility of the Fisher information matrix \( J_\omega \) for large \( M \). The proof is given in Appendix B.

**Proposition 4** For \( M \to \infty \) and QPSK modulation, the Fisher Information matrix \( J_\omega \) in (49) is invertible with probability one.

When the Fisher information matrix is invertible, the inverse can be calculated as follows [17]:

\[ J_{\omega}^{-1} = \frac{\eta}{2} \begin{bmatrix} J_{11}^{-1} + J_{11}^{-1} J_{12} D^{-1} J_{12}^T J_{11}^{-1} & -J_{11}^{-1} J_{12} D^{-1} \\ -D^{-1} J_{12}^T J_{11}^{-1} & D^{-1} \end{bmatrix}, \]  

(57)

with

\[ D = J_{22} - J_{12}^T J_{11}^{-1} J_{12} = \sum_{m=1}^{M} \left( J_{22}[m] - G[m] H[m]^{-1} G[m] \right). \]  

(58)
For large $M$, by the law of large numbers, we have

$$
\lim_{M \to \infty} \frac{2}{\eta M} D \xrightarrow{a.s.} \frac{2}{\eta} \left[ E \{ J_2 [m] \} - E \{ G[m] H[m]^{-1} G[m] \} \right] \triangleq J_\bar{S},
$$

(59)

where the expectation is with respect to the multiuser QPSK symbols $b[m]$.

Finally, let $\hat{w}_1$ be an unbiased estimate of the linear MMSE detector $w_1$ from the received signals $\{r[m]\}$ that makes use of the constant modulus property of the symbols, without knowing the signature waveforms $S$. Then the asymptotic Cramer-Rao bound ($K_w$ and $\bar{K}_w$) on $\hat{w}_1$ for large $M$ is given by (24) and (27), with the asymptotic Fisher information matrix $J_\bar{S}$ computed by (51)-(56) and (59), and the transformation matrix $H$ still given by (25) and (26).

5 Upper Bounds on SINR

Real-valued Signals: Let $\hat{w}_1$ be an estimate of the linear MMSE detector $w_1$. For BPSK modulation, the decision rule is $\hat{b}_1[m] = \text{sign} \left( \hat{w}_1^T r[m] \right)$. Note that the output of the estimated detector $\hat{w}_1$ can be written as $\hat{w}_1^T r[m] = w_1^T r[m] + (\hat{w}_1 - w_1)^T r[m]$. The first is the output of the exact linear MMSE detector, and the second term is a noise term due to the estimation error. The average output SINR of the estimated linear detector $\hat{w}_1$ is defined as [4]

$$
\text{SINR} \triangleq \frac{E \left\{ E \left\{ \hat{w}_1^T r[m] \mid b_1[m] \right\}^2 \right\}}{E \left\{ \text{Var} \left\{ \hat{w}_1^T r[m] \mid b_1[m] \right\} \right\}},
$$

(60)

The next result shows that the CRB on the estimated detector $\hat{w}_1$ can be translated into upper bound on the achievable SINR. The proof is given in Appendix C.

**Proposition 5** For the real-valued signal model (1) and BPSK modulation, assuming the estimated detector $\hat{w}_1$ is uncorrelated with a particular signal $r[m]$, then the following upper bound on the output SINR holds for any unbiased estimate $\hat{w}_1$ of the linear detector $w_1$:

$$
\text{SINR} \leq \frac{(w_1^T s_1)^2}{\sum_{k=2}^{K} (w_1^T s_k)^2 + \eta \|w_1\|^2 + \text{tr}(K_w C_r)},
$$

(61)

where $w_1$ is the exact linear MMSE detector given by (2), and $K_w$ is given by (5).
**Complex-valued Signals:** For BPSK modulation, \( b_k[m] \in \{+1, -1\} \), the decision rule on \( b_1[m] \) is \( \hat{b}_1[m] = \text{sign} \left[ \Re \left( \hat{w}_1^H r[m] \right) \right] \). The output SINR is defined as

\[
\text{SINR} = \frac{E \left\{ E \left\{ \Re \left( \hat{w}_1^H r[m] \right) \right\}^2 | b_1[m] \right\} - E \left\{ \text{Var} \left\{ \Re \left( \hat{w}_1^H r[m] \right) \right\} | b_1[m] \right\}}{E \left\{ \text{Var} \left\{ \Re \left( \hat{w}_1^H r[m] \right) \right\} | b_1[m] \right\}}.
\]

(62)

For QPSK modulation, \( b_k[m] \in \left\{ \frac{1+j}{\sqrt{2}}, \frac{-1+j}{\sqrt{2}}, \frac{1-j}{\sqrt{2}}, \frac{-1-j}{\sqrt{2}} \right\} \), the decision rule on \( b_1[i] \) is \( \hat{b}_1[m] = \frac{1}{\sqrt{2}} \text{sign} \left[ \Re \left( \hat{w}_1^H r[m] \right) \right] \), and \( \Im \hat{b}_1[m] = \frac{1}{\sqrt{2}} \text{sign} \left[ \Im \left( \hat{w}_1^H r[m] \right) \right] \). The output SINR is defined as

\[
\text{SINR} = \frac{E \left\{ E \left\{ \Re \left( \hat{w}_1^H r[m] \right) \right\} \Re \hat{b}_1[m] \right\}^2 - E \left\{ \text{Var} \left\{ \Re \left( \hat{w}_1^H r[m] \right) \right\} \Re \hat{b}_1[m] \right\} \}}{E \left\{ \text{Var} \left\{ \Re \left( \hat{w}_1^H r[m] \right) \right\} \Re \hat{b}_1[m] \right\} \}}.
\]

(63)

**Proposition 6** For the complex-valued signal model (1), assuming the estimated detector \( \hat{w}_1 \) is uncorrelated with a particular signal \( r[m] \), then the following upper bounds on the output SINR hold for any unbiased estimate \( \hat{w}_1 \) of the linear detector \( w_1 \). For BPSK modulation,

\[
\text{SINR} \leq \frac{(w_1^H s_1)^2}{\sum_{k=2}^{K} \left[ \Re \left\{ w_k^H s_k \right\} \right]^2 + \eta_1^2 \| w_1 \|^2 + \frac{1}{2} \text{tr} \left( K_w C_r + \Re \left\{ K_w C_r \right\} \right)};
\]

(64)

and for QPSK modulation,

\[
\text{SINR} \leq \frac{(w_1^H s_1)^2}{\sum_{k=2}^{K} \| w_k^H s_k \|^2 + \eta_1^2 \| w_1 \|^2 + \text{tr}(K_w C_r)};
\]

(65)

where \( w_1 \) is the exact linear MMSE detector given by (13), and \( K_w \) and \( K_r \) are given by (24) and (27).

6 **CRB for Detector Estimates in Unknown Multipath Channels**

In the preceding sections, it is assumed that the received signature waveforms \( \hat{S} \) of the first \( K \) users are known to the receiver. In multipath channels, the received signature waveform of each user is the convolution of the transmitted spreading sequence and the channel of that user, which can be expressed as \([20, 22]\) \( s_k = \Xi_k h_k \), \( k = 1, \cdots, K \), where \( h_k \in \mathbb{C}^L \) is the \( k \)-th user’s channel response, and \( \Xi_k \) is an \( N \times L \) matrix with each column consisting of the shifted spreading sequence of the \( k \)-th user. The received signal in (1) can be rewritten as

\[
r[m] = \sum_{k=1}^{K} \Xi_k h_k \hat{b}_k[m] + \sum_{k=1}^{K} s_k b_k[m] + n[m],
\]

(66)
In this section, it is assumed that the receiver has the knowledge of the spreading sequences of the first \( \tilde{K} \) users, and therefore \( \Xi_{1}, \ldots, \Xi_{K} \) are known. Hence there are \( (N\tilde{K} + L\tilde{K}) \) unknown complex parameters: \( \theta = [s_{T}^{T}, \ldots, s_{K}^{T}, h_{1}^{T}, \ldots, h_{\tilde{K}}^{T}]^{T} \). In what follows, we assume that the intersymbol interference is negligible so that the received signal vectors \( \{r[m]\} \) are i.i.d.

The linear MMSE detector \( w_{1} \) given by (13) can be rewritten as

\[
w_{1} = C_{r}^{-1}\Xi_{1}h_{1} = \left( \sum_{k=1}^{K} \Xi_{k}h_{k}h_{k}^{H}\Xi_{k}^{H} + SS^{H} + \eta I_{N} \right)^{-1}\Xi_{1}h_{1}.
\]

(67)

The difference between this case and the known channel case is that \( w_{1} \) is no longer uniquely identifiable from the observations: there is always an unknown phase ambiguity. This means that the Fisher information matrix for \( w_{1} \) is always singular, and therefore (5) no longer holds. The solution is to constrain \( w_{1} \). This can be most simply done by considering a modified detector defined by \( \tilde{w}_{1} = \varphi(w_{1})w_{1} \), where the possible form of \( \varphi(\cdot) \) includes

\[
\varphi(w_{1}) = w_{k}^{*}, \quad \text{or} \quad \varphi(w_{1}) = 1/w_{k}, \quad \text{or} \quad \varphi(w_{1}) = w_{k}^{*}/|w_{k}|.
\]

(68)

Let \( \hat{w}_{1} \) be an unbiased estimate of \( w_{1} \) based on \( M \) i.i.d. received signals \( \{r[m]\}_{m=1}^{M} \). Then the Cramer-Rao lower bound can be written as

\[
\text{Cov} \left\{ \left[ \mathbb{R}\hat{w} \, \Xi \hat{w} \right]^{T} \right\} \geq \frac{1}{M} \hat{\mathcal{H}}^{T} J_{\theta}^{\dagger} \hat{\mathcal{H}},
\]

(69)

where \( J_{\theta} \) is the Fisher information matrix, which will be specified later; and

\[
\hat{\mathcal{H}} \triangleq \frac{\partial \left[ \mathbb{R}\hat{w}_{1}^{T} \Xi \hat{w}_{1}^{T} \right]^{T}}{\partial \left[ \mathbb{R}\theta^{T} \Xi \theta^{T} \right]^{T}} = \frac{\partial \left[ \mathbb{R}\hat{w}_{1}^{T} \Xi \hat{w}_{1}^{T} \right]^{T}}{\partial \left[ \mathbb{R}\theta^{T} \Xi \theta^{T} \right]^{T}} \frac{\partial \left[ \mathbb{R}\theta^{T} \Xi \theta^{T} \right]^{T}}{\partial \left[ \mathbb{R}\hat{w}_{1}^{T} \Xi \hat{w}_{1}^{T} \right]^{T} \mathcal{H}}
\]

(70)

is a \( 2(N\tilde{K} + L\tilde{K}) \times 2N \) matrix. Note that the expressions for \( \partial w_{k}^{(y)}/\partial S_{i,j}^{(x)} \) are given by (25)-(26).

We next specify \( \partial w_{k}^{(y)}/\partial h_{i,j}^{(z)} \). From (67), the differential of \( w_{1} \) with respect to \( h_{k}, k = 1, \ldots, \tilde{K} \), is

\[
\Delta w_{1} = -C_{r}^{-1}\left( \Xi_{k}h_{k}\Delta h_{k}^{H} \Xi_{k}^{H} + \Xi_{k}\Delta h_{k}h_{k}^{H} \Xi_{k}^{H} \right) C_{r}^{-1}\Xi_{1}h_{1} + \delta_{k,1}C_{r}^{-1}\Xi_{1}\Delta h_{1}.
\]

(71)

Denote \( h_{i,j} \triangleq [h_{j}]_{i} \). We then have

\[
\frac{\partial w_{k}}{\partial h_{i,1}} = [C_{r}^{-1}\Xi_{1}]_{k,i} \left( 1 - h_{1}^{H}\Xi_{1}^{H} C_{r}^{-1}\Xi_{1}h_{1} \right),
\]

(72)

\[
\frac{\partial w_{k}}{\partial h_{i,j}} = -[C_{r}^{-1}\Xi_{j}]_{k,i} (h_{j}^{H}\Xi_{j}^{H} C_{r}^{-1}\Xi_{1}h_{1}), \quad j > 1,
\]

(73)

\[
\frac{\partial w_{k}}{\partial h_{i,j}^{*}} = -[C_{r}^{-1}\Xi_{j}h_{j}]_{k}[\Xi_{j}^{H} C_{r}^{-1}\Xi_{1}h_{1}], \quad j \geq 1.
\]

(74)
The SINR can be calculated similarly as before, e.g., Proposition 6. Clearly, for the known channel case, we can also use the transformation of parameters to obtain \( \tilde{w}_1 \) defined above. However note that in that case, the SINR of \( \hat{w}_1 \) is different from that of \( \hat{w}_1 \). The SINR calculated based on the decision statistic \( \hat{w}_1^H r[m] \) is, in fact, not a direct measure of the communication system performance, since it has to be combined with some kind of differential decoding procedure. However, it still gives an indication of performance, and in particular comparing the SINR’s of \( \hat{w}_1 \) for the known and unknown channel cases gives an indication of performance loss due to channel estimation. Thus, the significance of the result here is not on the absolute SINR, but rather on the SINR difference between the known and unknown channel cases. The most significant comparison is obtained when the influence of phase normalization \( \varphi(\cdot) \) on SINR is minimized. We found that the following \( \varphi(\cdot) \) has the least influence on the SINR:

\[
\varphi(w_1) = w^*_\kappa/|w_\kappa|, \quad \text{with } \kappa = \arg \max_k |w_k|.
\]  

For this transformation, we can calculate \( \tilde{H} \) in (70) according to the following steps: (i) Calculate \( \partial w_k / \partial S_{i,j} \) by (26), and calculate \( \partial w_k / \partial h_{i,j} \) by (72)-(74); (ii) For each parameter \( \theta_l \), apply the following transformation

\[
\frac{\partial \tilde{w}_k}{\partial \theta_l} = \frac{w^*_\kappa}{|w_\kappa|} \left( \frac{\partial w_k}{\partial \theta_l} \right)^* + \frac{w_k}{|w_\kappa|^2} \left[ \frac{1}{|w_\kappa|^2} \left( \frac{\partial w_k}{\partial \theta_l} \right)^* - \left( \frac{w^*_\kappa}{|w_\kappa|^2} \right) \left( \frac{\partial w_k}{\partial \theta_l} \right) \right]
\]  

(76)

\[
\frac{\partial \tilde{w}_k}{\partial \theta_l} = \frac{w^*_\kappa}{|w_\kappa|} \left( \frac{\partial w_k}{\partial \theta_l} \right)^* + \frac{w_k}{|w_\kappa|^2} \left[ \frac{1}{|w_\kappa|^2} \left( \frac{\partial w_k}{\partial \theta_l} \right)^* - \left( \frac{w^*_\kappa}{|w_\kappa|^2} \right) \left( \frac{\partial w_k}{\partial \theta_l} \right) \right]
\]  

(77)

(iii) Apply the transformation (25) to get the real-valued transformation \( \tilde{H} \) appeared in (69).

We next specify the Fisher information matrix \( J_\theta \) for each method when the channels are unknown.

**FA Case:** The Fisher information matrix \( J_\theta \) for \( [\Re(\Theta) \Im(\Theta)] \) is a \( 2(N\tilde{K} + L\tilde{K}) \times 2(N\tilde{K} + L\tilde{K}) \) matrix whose elements are given by

\[
J_{u,v}^{x,y} = E \left\{ \frac{1}{f(r;\theta)} \left( \frac{\partial f(r;\theta)}{\partial \theta_u^{[x]}} \right)^* \left( \frac{\partial f(r;\theta)}{\partial \theta_v^{[y]}} \right) \right\}
\]  

(78)

Note that \( \partial f(r;\theta)/\partial S_{i,j}^{[x]} \) is given by (18). In addition, we have for \( x \in \{r, i\} \)

\[
\frac{\partial f(r;\theta)}{\partial h_{i,j}^{[x]}} = \frac{2}{2^{|\eta\eta|}} \sum_n \sum_b \sum_{\xi_m} \left( r_n - \xi_m^H b \right)^{[x]} \left[ \Re(\Xi_{j,n,i}^b) - \Im(\Xi_{j,n,i}^b) \right]
\]  

\[
+ \left[ -j (r_n - \xi_m^H b)^{[x]} \left[ \Re(\Xi_{j,n,i}^b + \Im(\Xi_{j,n,i}^b) r - Sb) \right] \right] \psi_\eta(||r - Sb||),
\]  

(79)
where $\xi^H_i$ denotes the $i$-th row of $S$.

Denote $R_j \triangleq \Re(\Xi_j^H \Xi_j)$. Then we have the following result on the asymptotic Fisher information matrix at high SNR. The proof is given in Appendix D.

**Proposition 7** The Fisher information matrix $J_S$ given by (78) has the following limit

$$
\lim_{\eta \to 0} \eta J_{u,v}^{x,y} = \begin{cases} 
2 \delta_{i,k} \delta_{j,l} \delta_{x,y}, & \text{if } \theta_u = \tilde{S}_{i,j} \text{ and } \theta_v = \tilde{S}_{k,l} \\
2 \delta_{j,l} \delta_{x,y} [R_j]_{i,k}, & \text{if } \theta_u = h_{i,j} \text{ and } \theta_v = h_{k,l} \\
0, & \text{if } \theta_u = S_{i,j} \text{ and } \theta_v = h_{k,l} \text{ or if } \theta_u = h_{i,j} \text{ and } \theta_v = S_{k,l}
\end{cases} \quad (80)
$$

or in matrix form

$$
\lim_{\eta \to 0} \eta J_{\theta} = 2 \text{diag} \left[ I_{N \tilde{K}}, R_1, \ldots, R_{\tilde{K}}, I_{N \tilde{K}}, R_1, \ldots, R_{\tilde{K}} \right]. \quad (81)
$$

**SO Case:** In this case, the matrix $J_S$ is specified by

$$
J_{u,v}^{x,y} = \frac{\partial x^T}{\partial \theta^{|x|}} \Sigma^{-1} \frac{\partial x}{\partial \theta^{|y|}}, \quad (82)
$$

where $\Sigma$ is specified by (38)-(40); and $\partial x^T / \partial \theta^{|x|}$ is specified by (41) and (42) when $\theta_u = S_{i,j}$.

When $\theta_u = h_{i,j}$, $\partial x^T / \partial \theta^{|x|}$ can be obtained by using the following derivatives

$$
\frac{\partial C_{m,n}}{\partial h_{i,j}} = [\Xi]_{m,i}[\xi_j^H \xi_j^H]_n, \quad \text{and} \quad \frac{\partial C_{m,n}}{\partial h_{i,j}^*} = [\Xi]_{j,m}[\Xi_j^H]_n, \quad (83)
$$

and the similar transformation as (42).

**CM Case:** Here to resolve the possible phase ambiguity between $\tilde{h}_k$ and $\tilde{b}_k$, as well as that between $\tilde{s}_k$ and $\tilde{b}_k$, we assume that $b[1]$ is known. Then the unknown parameters are

$$
\omega \triangleq [\phi[2]^T \ldots \phi[M]^T \text{vec}(\Re(\theta)^T \text{vec}(\Re(\theta)^T]^T. \quad (84)
$$

Denote $\tilde{B}[m] \triangleq \text{diag}(b_1[m], \ldots, b_1[m], b_2[m], \ldots, b_2[m], \ldots, b_K[m], \ldots, b_K[m]), B[m] \triangleq \tilde{b}[m]^T \otimes I_N$, and $\hat{\Xi} \triangleq [\Xi_1 \ldots \Xi_K]$. Then the received signal can be rewritten as

$$
\tilde{r}[m] = [\hat{\Xi} \tilde{B}[m], \tilde{J} \hat{\Xi} \tilde{B}[m]] \begin{bmatrix} \text{vec}(\Re \hat{H}) \\ \text{vec}(\Im \hat{H}) \end{bmatrix} + [B[m], J \tilde{B}[m]] \begin{bmatrix} \text{vec}(\Re \hat{S}) \\ \text{vec}(\Im \hat{S}) \end{bmatrix} + n[m]. \quad (85)
$$
The Fisher information matrix is still given by (47), with

\[
\frac{\partial \mu}{\partial \omega} = \begin{bmatrix}
0 & \cdots & 0 & j\tilde{B}[1] & \tilde{B}[1] & j\tilde{B}[1] & j\tilde{B}[1] \\
\vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & j\beta[M] & \tilde{B}[M] & \tilde{B}[M] & j\tilde{B}[M] & j\tilde{B}[M]
\end{bmatrix}.
\]

Accordingly, the asymptotic Fisher information matrix \( J_\theta \) for large \( M \) is given by

\[
J_\theta \approx \frac{2}{\eta} \left[ E \{ J_{22}[m] \} - E \{ G[m]^T H[m]^{-1} G[m] \} \right],
\]

with

\[
H[m] = \Re (\beta[m]^H S^H \beta[m]),
\]

\[
J_{22}[m] = \Re \left\{ \left[ \tilde{B}[m], \tilde{B}[m], j\tilde{B}[m], j\tilde{B}[m] \right]^H \left[ \tilde{B}[m], \tilde{B}[m], j\tilde{B}[m], j\tilde{B}[m] \right] \right\},
\]

\[
G[m] = \Re \left\{ -j\beta[m]^H S^H \left[ \tilde{B}[m], \tilde{B}[m], j\tilde{B}[m], j\tilde{B}[m] \right] \right\}.
\]

7 Numerical Results

In this section, we provide some numerical results to compare the SINR bounds of different schemes derived in this paper. We will also compare the SO bounds with the actual SINR expressions of various blind and group-blind linear detectors obtained in [4]. All the results are for complex signals and QPSK modulation, and we only illustrate their respective asymptotic bounds.

Systems with Equicorrelated Waveforms and Known Channels

We first evaluate the SINR bounds of various schemes in a system of \( K \) users with equicorrelated signature waveforms, i.e., \( s_i^T s_j = \rho \) for all \( i \neq j \), where \( 0 \leq \rho < 1 \), and \( \| s_k \| = 1, \forall k \). Such a system although simple, has the advantage of being completely specified by the single parameter \( \rho \).

Note that it is easy to see that the performance bounds for the known channel case (Sections 2-4) is a function of the user spreading waveforms \( S \) only through their correlation matrix \( R = S^H S \), because the performance of linear detectors is invariant to orthogonal coordinate transformations.

For the equicorrelated system, we have \( R = (1 - \rho)I_K + \rho 1_K 1_K^T \), where \( 1_K \) denotes a \( K \)-vector of all 1’s. Given \( R \), we can for example designate \( S \) to be of the form \( S = \left[ 0_{K \times (N-K)} \sqrt{R}^T \right]^T \) (where \( \sqrt{R} \) denotes the Cholesky factor of \( R \)), and then evaluate the various SINR bounds.
Figure 1: Numerical results for equicorrelated system. $N = 13$, $K = 10$, $\tilde{K} = 5$. All users have equal powers and $\text{SNR} = \frac{1}{\eta} = 16\text{dB}$. The top thick lines represent the performance of the exact linear MMSE detector.

Figure 2: Numerical results for systems with random waveforms. $N = 13$, $K = 10$, $\tilde{K} = 5$, $\text{SNR} = \frac{1}{\eta} = 16\text{dB}$. The top thick lines represent the performance of the exact linear MMSE detector. (Left: median; Right: 10-percentile.)
In Figures 1 (a) & (b) we illustrate the various SINR bounds for an equicorrelated system as a function of \( \rho \) and \( M \), respectively. The parameters are \( N = 13, K = 10, \hat{K} = 5, \) SNR = \( \frac{1}{\eta} = 16 \)dB. We have calculated the SINR bounds for both \( \hat{K} = 1 \) (blind detector) and \( \hat{K} = 5 \) (group-blind detector) for the three schemes. These two bounds under both the FA and the CM schemes are too close to be distinguished in the scale of the plot, whereas under the SO scheme there is a gap in between them. In the same figures, we also plot the actual SINR values for three second-order-moments-based detectors, namely, the direct-matrix-inversion (DMI) blind detector, the subspace-blind detector, and the group-blind detector, based on the analytical SINR expressions given in [4]. The SINR values of the exact linear MMSE detector is also shown. Several observations are made from these figures. First, as expected, the SINR upper bounds are ordered starting from the best as the FA detector, the CM detector, and the SO detector. Secondly, the SINR bound for the FA detector and that for the CM detector are quite close, and both are close to the SINR of the exact linear MMSE detector; whereas the SINR bound for the SO detector is fairly away from those of the FA and the CM detectors. Thirdly, the performance of the subspace detector is very close to the SO bound - indicating that the subspace-based detector is near-optimal among the class of SO detectors; whereas the DMI detector is significantly away from the SO bound. Moreover, it is seen from Figure 1 (b) that these observations hold true for all ranges of \( \rho \).

**Systems with Random Waveforms and Known Channels**

Next we evaluate the SINR bounds for various schemes with randomly generated signature waveforms. The parameters are \( N = 13, K = 10, \) SNR = \( \frac{1}{\eta} = 16 \)dB. 1000 sets of randomly signature waveforms are independently generated. For each set of waveforms, we calculate the corresponding SINR bounds for both \( \hat{K} = 1 \) (blind detector) and \( \hat{K} = 5 \) (group-blind detector), as well as the actual SINR’s for the three detectors mentioned above. In Figures 2 (a) & (b), we plot the median and 10-percentile (i.e., “worst case”) results. It is seen that the observations made in the equicorrelated systems still hold in systems with random signature waveforms. Specifically, the CM bound seems to be even closer to the FA bound, and the gap between the FA/CM bound and the SO bound is even larger. The SINR of the group-blind detector does not quite reach the bound for the 10-percentile, particularly for large \( M \). This is due to the fact that the exact group-blind detector does not correspond to the linear MMSE detector, but rather to a linear hybrid
(zero-forcing/MMSE) detector [20].

Figure 3: Histogram of SINR loss in dB for systems with random waveforms. $N = 13$, $\tilde{K} = K = 5$, $\text{SNR} = \frac{1}{\eta} = 16\text{dB}$, and $M = 80$. The bottom figure is a close-up of the top figure. The bounds are shown with solid lines, the actual SINR of the blind and group-blind algorithms in [20, 23] with dashed lines.

In Figure 3, we give the histograms of the SINR loss in dB for various schemes compared with the SINR of the exact linear MMSE detector, for $M = 80$. Interestingly, it is seen that the performance loss incurred by the FA and CM detectors is almost invariant to the choice of the signature waveforms; whereas the performance of the SO detectors varies significantly with different set of waveforms. Moreover, again it is seen that FA and CM detectors perform quite closely, and both substantially outperform the SO detectors. The figure also shows the advantage of using group-blind algorithms rather than blind algorithms.

Finally, Figures 4 (a) & (b) give the same results for a larger spreading gain and a near-far situation. Here $N = 31$, $K = 16$, $\tilde{K} = 8$, $\text{SNR} = \frac{1}{\eta} = 16\text{dB}$. All the known users have power $1\text{ dB}$, while four of the unknown users have power $-10\text{dB}$ and the other four have power $+20\text{dB}$.
Figure 4: Numerical results for systems with random waveforms and near-far situation. $N = 31$, $K = 16$, $\tilde{K} = 8$, SNR = $\frac{1}{\eta} = 16$dB. All the known users have power 1 dB, while four of the unknown users have power $-10$dB and the other four have power $+20$dB. The top thick lines represent the performance of the exact linear MMSE detector. (Left: median; Right: 10-percentile.)

Figure 5: SINR bounds for the unknown channel case with spreading gain 10, $\tilde{K} = \tilde{K} = 4$ users with equal powers, channel length $L = 6$, and SNR = $\frac{1}{\eta} = 16$dB. The top thick lines represent the performance of the exact linear MMSE detector. (Left: median; Right: 10-percentile.)
Systems with Random Signature Waveforms and Unknown Channels

We next illustrate the various SINR bounds in systems with random signature waveforms and unknown multipath channels. We assume that the user spreading codes are randomly generated with spreading gain 10. There are \( K = 8 \) users, and \( \tilde{K} = \bar{K} = 4 \). The number of channel coefficients is \( L = 6 \). For simplicity, a guard interval of length 6 chip intervals is inserted between symbols to avoid intersymbol interference [8, 21]. Hence the length of the signature waveform vector is \( N = 15 \). The channel coefficients \( h_k \) are generated according a complex Gaussian distribution and then normalized such that \( h_k \) has a unit norm. The median and 10-percentile performance of various methods over 1000 realizations of random codes and channels are shown in Figures 5 (a) & (b), where performance under known and unknown channels is compared. From these figures, it is seen that the performance (bound) loss due to unknown channels is quite insignificant. In Figure 6, we plot the histogram of the difference in dB between the SINR bounds for known and unknown channel cases, for FA, CM and SO detectors (\( M = 80 \)). It is seen that the loss due to unknown channels is a fractional of a dB for both FA and CM detectors, and it is bigger for the SO detectors (up to 1dB).

![Figure 6: Histogram of SINR loss in dB due to unknown channel with spreading gain 10, \( \tilde{K} = \bar{K} = 4 \) users with equal power, channel length \( L = 6 \), and SNR = \( \frac{1}{M} = 16 \text{dB} \), and \( M = 80 \).]
8 Conclusions

We have derived upper bounds for the SINR of estimated blind or group-blind linear multiuser detectors under three different assumptions on the prior information that the receiver can exploit, namely, (i) the finite-alphabet (FA) property of the symbols; (ii) the constant-modulus (CM) property of the symbols; and (ii) the second-order (SO) moments of the received signals. The numerical results obtained reveal that the SINR bounds for both the FA and the CM detectors are fairly close to the SINR of the exact linear MMSE detector; whereas there is a nontrivial gap between the SINR bounds of the FA/CM detectors and that of the SO detector. These results show that a potential gain can be obtained by exploiting more structural information of the system (e.g., FA or CM) rather than only the second-order statistics. Some related work along this line includes [5, 9, 10, 19, 24]. We have also seen that the SO bound is quite tight, since it is actually achieved by the subspace-based blind or group-blind algorithms in [20, 23]. Moreover, it is seen that under both FA and CM schemes, the bounds for blind and group-blind detectors are virtually indistinguishable, suggesting that by “optimal” processing, the lack of information about the signature waveforms of interfering users will result in little performance loss. Finally, we remark that the analysis developed in this paper applies to multiple-antenna communication systems as well, where the signature waveforms $s_k$ become the composite effects of the spreading codes and the spatial channels [13].

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Appendix A: Proof of Propositions 1 and 2

We will make use of the following lemma. Let $x \sim N(\mu, \eta I_N)$. Denote the pdf of $x$ as

$$
\phi_\eta(\|x - \mu\|) \triangleq (2\pi\eta)^{-\frac{N}{2}} \exp\left(-\frac{\|x - \mu\|^2}{2\eta}\right). \quad (91)
$$
Lemma 1 For any $\delta > 0$, we have

$$\lim_{\eta \to 0} \frac{1}{\eta} \int_{|x-\mu|>\delta}(x_i-\mu_i)^2\phi_\eta(||x-\mu||)dx = 0, \quad (92)$$

and

$$\lim_{\eta \to 0} \frac{1}{\eta} \int_{|x-\mu|<\delta}(x_i-\mu_i)^2\phi_\eta(||x-\mu||)dx = 1. \quad (93)$$

Proof: Denote $\delta' = \frac{4}{\sqrt{N}}$. First note that

$$\int_{|x-\mu|>\delta}(x_i-\mu_i)^2\phi_\eta(||x-\mu||)dx \leq \sum_{j=1}^{N} \int_{|x_j-\mu_j|>\delta'}(x_i-\mu_i)^2\phi_\eta(||x-\mu||)dx, \quad (94)$$

and

$$\int_{|x_i-\mu_i|>\delta'}(x_i-\mu_i)^2\phi_\eta(||x-\mu||)dx = \int_{|x_i-\mu_i|>\delta'}(x_i-\mu_i)^2 \frac{1}{\sqrt{2\pi\eta}} \exp\left[-\frac{(x_i-\mu_i)^2}{2\eta}\right] dx_i. \quad (95)$$

Without loss of generality, let $\mu_i = 0$. Using integral-by-part, we have

$$\frac{1}{\eta} \int_{\delta'}^\infty x^2 \frac{1}{\sqrt{2\pi\eta}} \exp\left(-\frac{x^2}{2\eta}\right) dx = \frac{\delta'}{\sqrt{2\pi\eta}} \exp\left(-\frac{\delta'^2}{2\eta}\right) + Q\left(\frac{\delta'}{\eta}\right) \to 0, \text{ as } \eta \to 0. \quad (96)$$

Hence the term in the sum corresponding to $i = j$ in (94) converges towards zero. For the terms with $i \neq j$ we have as $\eta \to 0$

$$\frac{1}{\eta} \int_{|x_j-\mu_j|>\delta'}(x_i-\mu_i)^2\phi_\eta(||x-\mu||)dx = \int_{|x_j-\mu_j|>\delta'} \frac{1}{\sqrt{2\pi\eta}} \exp\left[-\frac{(x_j-\mu_j)^2}{2\eta}\right] dx_j \to 0. \quad (97)$$

Hence (92) holds. Moreover since for $x \sim N(\mu, \eta I_N)$, $E\{(x_i-\mu_i)^2\} = \int (x_i-\mu_i)^2\phi_\eta(||x-\mu||)dx = \eta$, then this and (92) imply (93).

Proof of Proposition 1: Denote $\alpha(r) \triangleq [\sum_b \phi_\eta(||r-Sb||)]^{-1}$. Define

$$h^{i,k}_\eta(r; b^m, b^n) \triangleq \frac{\alpha(r)}{\eta} (r_i - \xi_i^T b^m) (r_k - \xi_k^T b^n) \phi_\eta(||r-Sb^m||) \phi_\eta(||r-Sb^n||). \quad (98)$$

Then using (8), we have

$$\eta J_{(i,j)(k,l)} = \frac{1}{2\kappa} \sum_m \sum_n \tilde{u}_{ijm} \tilde{q}_{ikn} \int_{\mathbb{R}^N} h_\eta(r; b^m, b^n) dr. \quad (99)$$

First we show that if $b_m \neq b_n$, then $\int h_\eta(r; b^m, b^n) dr \to 0$. Define the half space $\Gamma \triangleq \{ r \in \mathbb{R}^N : ||r-Sb^m|| \geq ||r-Sb^n|| \}$. Then for any $r \in \Gamma$,

$$|h^{i,k}_\eta(r; b^m, b^n)| \leq \frac{1}{\eta} \phi_\eta(||r-Sb^n||^{-1} |r_i - \xi_i^T b^m| |r_k - \xi_k^T b^n| \phi_\eta(||r-Sb^m||) \phi_\eta(||r-Sb^n||) \leq \frac{1}{2\eta} \left[ (r_i - \xi_i^T b^m)^2 + (r_k - \xi_k^T b^n)^2 \right] \phi_\eta(||r-Sb^m||). \quad (100)$$
Similarly as above we have
\[
\int_{\Gamma} |h_{\eta}^{i,k}(r; b^m, b^n)| dr \leq \frac{1}{\eta} \int_{\Gamma} (r_i - \xi_i T b^m) \phi_\eta(||r - Sb^m||) dr + \frac{1}{\eta} \int_{\Gamma} (r_k - \xi_k T b^n) \phi_\eta(||r - Sb^m||) dr \to (101)
\]
as \(\eta \to 0\), by Lemma 1, since by the definition of \(\Gamma\), we have \(||r - Sb^m|| \geq \frac{1}{2} ||Sb^n - Sb^m|| > 0\).
Similarly, \(\int_{\mathbb{R}^N \backslash \Gamma} |h_{\eta}^{i,k}(r; b^m, b^n)| dr \to 0\). Hence \(\int_{\mathbb{R}^N} h_{\eta}^{i,k}(r; b^m, b^n) dr \to 0\) for \(b^m \neq b^n\).

Next we show that if \(b_m = b_n\), then \(\int h_{\eta}^{i,k}(r; b^m, b^n) dr \to \delta_{i,k}\). Define a ball \(C\) centered at \(b^m\) which is sufficiently small, such that
\[
\forall r \in C : ||r - Sb|| > ||r - Sb^m||, \text{ for any } b \in \{+1, -1\}^K \text{ and } b \neq b^m. \tag{102}
\]
Similarly as above we have
\[
\int_{\mathbb{R}^N \backslash C} h_{\eta}^{i,k}(r; b^m, b^n) dr \to 0, \quad \text{as } \eta \to 0. \tag{103}
\]
Now since \(\frac{\phi_\eta(||r - Sb||)}{\phi_\eta(||r - Sb^m||)} = \exp\left(-\frac{||r - Sb||^2 - ||r - Sb^m||^2}{2\eta}\right)\), then by (102) we have
\[
\forall r \in C \forall b \neq b^m \lim_{\eta \to 0} \frac{\phi_\eta(||r - Sb||)}{\phi_\eta(||r - Sb^m||)} = 0. \tag{104}
\]
For any \(r \in C\), using (98) we have
\[
\left|h_{\eta}^{i,k}(r; b^m, b^n) - \frac{1}{\eta} (r_i - \xi_i T b^m) (r_k - \xi_k T b^n) \phi_\eta(||r - Sb^m||)\right|
= \left|\frac{1}{\eta} (r_i - \xi_i T b^m) (r_k - \xi_k T b^n) \phi_\eta(||r - Sb^m||)\right|
\left[1 - \left[1 + \sum_{b \neq b^m} \frac{\phi_\eta(||r - Sb||)}{\phi_\eta(||r - Sb^m||)}\right]^{-1}\right]. \tag{105}
\]
where \(\lim_{\eta \to 0} \epsilon(\eta) = 0\) by (104). Hence we have
\[
\left|\int_C h_{\eta}^{i,k}(r; b^m, b^n) - \frac{1}{\eta} (r_i - \xi_i T b^m) (r_k - \xi_k T b^n) \phi_\eta(||r - Sb^m||) dr\right|
\leq \int_C \left|h_{\eta}^{i,k}(r; b^m, b^n) - \frac{1}{\eta} (r_i - \xi_i T b^m) (r_k - \xi_k T b^n) \phi_\eta(||r - Sb^m||)\right| dr
\leq \epsilon(\eta) \cdot \frac{1}{\eta} \int_C |r_i - \xi_i T b^m||r_k - \xi_k T b^n| \phi_\eta(||r - Sb^m||) dr \to 0, \tag{106}
\]
as \(\eta \to 0\), since by Lemma 1 the limit of \(\frac{1}{\eta} \int_C\) in (106) is bounded. Therefore
\[
\lim_{\eta \to 0} \frac{1}{\eta} \int_C h_{\eta}^{i,k}(r; b^m, b^n) dr = \lim_{\eta \to 0} \frac{1}{\eta} \int_C (r_i - \xi_i T b^m) (r_k - \xi_k T b^n) \phi_\eta(||r - Sb^m||) dr = \delta_{i,k}, \tag{107}
\]
where the last equality is obtained by Lemma 1 for \(i = k\), and for \(i \neq k\) from the fact that \(C\) is symmetric and the integral is therefore 0 for all values of \(\eta\). Combining (103) and (107), we obtain
\[
\int h_{\eta}^{x,k}(r; b^m, b^n)dr \to \delta_{i,k}.
\]
In summary, we have \(\int_{\mathbb{R}^N} h_{\eta}^{x,k}(r; b^m, b^n)dr \to \delta_{i,k}\delta_{m,n}\). Substituting this into (99), we obtain Proposition 1.

**Proof of Proposition 2:** We will make use of the following lemma. Let \(x \sim \mathcal{N}_c(\mu, \eta I_N)\). Denote the pdf of \(x\) as \(\psi_\eta(\|x - \mu\|) \triangleq (\pi\eta)^{-N} \exp \left( - \frac{\|x - \mu\|^2}{\eta} \right)\).

**Lemma 2** For any \(\delta > 0\), and \(u \in \{r,i\}\), we have
\[
\lim_{\eta \to 0} \frac{2}{\eta} \int_{\|x - \mu\| > \delta} \left( x_i - \mu_i \right)^2 \phi_\eta(\|x - \mu\|)dx = 0, \tag{108}
\]
and
\[
\lim_{\eta \to 0} \frac{2}{\eta} \int_{\|x - \mu\| < \delta} \left( x_i - \mu_i \right)^2 \phi_\eta(\|x - \mu\|)dx = 1. \tag{109}
\]

**BPSK modulation:** In this case, \(\mathbb{R}b_j = \mathbb{b}_j \in \{+1, -1\}\), and \(\mathcal{N}b_j = 0\). Define
\[
h_{\eta}^{x,y}(r; b^m, b^n) \triangleq \frac{2\alpha(r)}{\eta} \left( r_i - \xi_i^T b^m \right)^{(x)} \left( r_k - \xi_i^T b^n \right)^{(y)} \psi_\eta(\|r - Sb^m\|) \psi_\eta(\|r - Sb^n\|),
\]
with \(\alpha(r) \triangleq \left[ \sum b \psi_\eta(\|r - Sb\|) \right]^{-1}\). Then using (15) and (18), we have
\[
\eta J^{x,y}_{(i,j)(k,l)} = \frac{2\alpha}{\eta} \sum_m \sum_n b_m^i b_l^j \int_{\mathbb{C}^N} h_{\eta}^{x,y}(r; b^m, b^n)dr. \tag{110}
\]
Following the same line of proof as before, it can be shown that if \(b_m \neq b_n\), then \(\int h_{\eta}^{x,y}(r; b^m, b^n)dr \to 0\); and if \(b_m = b_n\), then \(\int h_{\eta}^{x,y}(r; b^m, b^n)dr \to \delta_{i,k}\delta_{x,y}\). Substituting these into (110), we obtain
\[
\eta J^{x,y}_{(i,j)(k,l)} \to \delta_{i,k}\delta_{j,x,y}.
\]

**QPSK modulation:** In this case, \(\mathbb{R}b_j \) and \(\mathcal{N}b_j \) take values from \(\{+\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\}\). Define
\[
h_{\eta}^{x,y}(r; b^m, b^n) \triangleq \frac{2\alpha(r)}{\eta} \left( r_i - \xi_i^T b^m \right)^{(x)} \left( r_k - \xi_i^T b^n \right)^{(y)} \psi_\eta(\|r - Sb^m\|) \psi_\eta(\|r - Sb^n\|), \tag{111}
\]
\[
h_{\eta}^{x,y}(r; b^m, b^n) \triangleq \frac{2\alpha(r)}{\eta} \left( r_i - \xi_i^T b^m \right)^{(x)} \left[ -j \left( r_k - \xi_i^T b^n \right)^{(y)} \psi_\eta(\|r - Sb^m\|) \psi_\eta(\|r - Sb^n\|), \tag{112}
\]
\[
h_{\eta}^{x,y}(r; b^m, b^n) \triangleq \frac{2\alpha(r)}{\eta} \left[ -j \left( r_i - \xi_i^T b^n \right)^{(y)} \psi_\eta(\|r - Sb^m\|) \psi_\eta(\|r - Sb^n\|) \tag{113}
\]
\[
h_{\eta}^{x,y}(r; b^m, b^n) \triangleq \frac{2\alpha(r)}{\eta} \left[ -j \left( r_i - \xi_i^T b^n \right)^{(y)} \psi_\eta(\|r - Sb^m\|) \psi_\eta(\|r - Sb^n\|). \tag{114}
\]

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Then using (15) and (18), we have

$$\eta J^{xy}_{(i,j)(k,l)} = \frac{2}{q^2} \sum_{m} \sum_{n} \left[ \Re \hat{b}_{m} \Re \hat{b}_{n} \int_{CN} h_{\eta,rr'}(r; b^m, b^n) \, dr + \Re \hat{b}_{m} \Im \hat{b}_{n} \int_{CN} h_{\eta,rr'}(r; b^m, b^n) \, dr \right] \frac{\partial h}{\partial x} \frac{\partial h^T}{\partial x} \cdot (115)$$

Following the same line of proof as before, it can be shown that

$$\int h_{\eta,uu}(r; b^m, b^n) \, dr \rightarrow \delta_{i,k} \delta_{x,y} \delta_{m,n} \delta_{u,v},$$

where $u,v \in \{ r, i \}$. Substituting this into (110), we obtain $\eta J^{xy}_{(i,j)(k,l)} \rightarrow \delta_{i,k} \delta_{j,l} \delta_{x,y}$.

**Appendix B: Proof of Propositions 3 and 4**

**Proof of Proposition 3:** By slightly modifying the proof of Theorem 1 in [12] we have

$$\lim_{M \to \infty} M \cdot E \{ (\alpha - \alpha)(\dot{\alpha} - \alpha)^T \} = \frac{\partial h}{\partial x} \Sigma \frac{\partial h^T}{\partial x}. \quad (116)$$

Define $P \triangleq \left[ \begin{array}{c} \frac{\partial h}{\partial x} \Sigma^{\frac{1}{2}} \\
\frac{\partial f}{\partial \theta} \Sigma^{-\frac{1}{2}} \end{array} \right]$. Then we have $0 \leq PP^T = \left[ \begin{array}{cc} \frac{\partial h}{\partial x} \Sigma \frac{\partial h^T}{\partial x} & \frac{\partial g}{\partial \theta} \\
\frac{\partial f^T}{\partial \theta} \Sigma^{-1} & \frac{\partial f^T}{\partial \theta} \right]$, since

$$\frac{\partial g}{\partial \theta} = \frac{\partial h}{\partial x} \frac{\partial f}{\partial \theta}. \quad \text{From this it then follows that} \quad (117)$$

$$\frac{\partial h}{\partial x} \Sigma \frac{\partial h^T}{\partial x} \geq \frac{\partial g}{\partial \theta} \left( \frac{\partial f^T}{\partial \theta} \Sigma^{-1} \frac{\partial f}{\partial \theta} \right)^T \frac{\partial g}{\partial \theta}. \quad (117)$$

**Proof of Proposition 4:** The proof becomes simpler in notation if we take the QPSK constellation as $\{1, -1, j, -j\}$, which we can do since the proposition does not depend on the particular QPSK constellation.

We need to show that $\Re \left\{ \frac{\partial \bar{u}}{\partial \omega} \right\}^T \Im \left\{ \frac{\partial \bar{u}}{\partial \omega} \right\}^T$ has full column rank as $M \to \infty$. This is equivalent to showing that there exists no real vector $v \neq 0$ such that $\frac{\partial \bar{u}}{\partial \omega} v = 0$. Note that any real linear combination of the columns in $[\bar{B}[m] \quad j\bar{B}[m]]$ can be written as $\bar{V} \bar{b}[m]$ for some complex matrix $\bar{V}$. Then using (48) we can write $\frac{\partial \bar{u}}{\partial \omega} v = 0. \quad \text{as}$

$$\tilde{S} \tilde{A}_1 \bar{b}[1] = \tilde{V} \bar{b}[1], \quad (118)$$

$$S \tilde{A}_1 \bar{b}[m] = \tilde{V} \bar{b}[m], \quad m = 2, \ldots, M, \quad (119)$$

where $\tilde{A}_i$ is a diagonal matrix with real elements, and $\bar{V} = -j\tilde{V}$. Write (119) as

$$A_i \bar{b}[m] = \underbrace{(S^H S)^{-1} S^H \tilde{V} \bar{b}[m]}_{\bar{A}}, \quad m = 2, \ldots, M, \quad (120)$$
Next we show that $A_{kl} = A_{kk}\delta_{kl}$ as $M \to \infty$. Otherwise suppose that for some $k$, there are at least two non-zero $A_{kl}$’s. Then as $M \to \infty$, with probability one, there exists some $m$ and $\tilde{b}_1[m], \ldots, \tilde{b}_K[m]$, so that the right-hand side of (121) is neither purely real nor purely imaginary. On the other hand, the left-hand side of (121) is always purely real or imaginary (since $A_{kk}$ is real). Thus, in order for (121) to be consistent for any $M$, only one of $A_{kl}$ can be non-zero for any $k = 1, \ldots, K$; and moreover, it is easily seen that this non-zero element must be $A_{kk}$.

From the definition of $A$ in (120), it then follows that $V = \tilde{S}\text{diag}(A)$. However then (118) cannot be satisfied, since its right-hand side is a non-zero vector in span($\tilde{S}$), and its left-hand side is in span($\tilde{S}$), but $S = \begin{bmatrix} \tilde{S} & \bar{S} \end{bmatrix}$ has full column rank. Therefore $\begin{bmatrix} \Re \left\{ \frac{\partial H}{\partial \omega} \right\}^T & \Im \left\{ \frac{\partial H}{\partial \omega} \right\}^T \end{bmatrix}^T$ must have full column rank. 

\[ \Rightarrow \quad A_{kk}b_k[m] = \sum_{i=1}^{K} A_{ki}\tilde{b}_i[m], \quad m = 2, \ldots, M; \quad k = 1, \ldots, K. \tag{121} \]

Proof of Proposition 6: Denote $\Delta w_1 \triangleq \hat{w}_1 - w_1$. The numerator and the denominator of (60) can be calculated as follows:

\begin{align*}
E \left\{ \hat{w}_1^T r[m] \mid b_1[m] \right\} &= b_1[m] (w_1^T s_1), \tag{122} \\
\text{Var} \left\{ \hat{w}_1^T r[m] \mid b_1[m] \right\} &= \sum_{k=2}^{K} (w_1^T s_k)^2 + \eta\|w_1\|^2 + E \left\{ (\Delta w_1^T r[m])^2 \right\}, \tag{123} \\
\text{where } E \left\{ (\Delta w_1^T r[m])^2 \right\} &= \text{tr} \left( E \left\{ \Delta w_1 \Delta w_1^T \right\} E \left\{ r[m]r[m]^T \right\} \right) \geq \text{tr} \left( K_w C_r \right), \tag{124}
\end{align*}

where (124) follows from (5). Substituting (122)-(124) into (60), we obtain (61).

Proof of Proposition 6: For BPSK modulation, the numerator and the denominator of (62) can be calculated as follows:

\begin{align*}
E \left\{ \Re \left( w_1^H r[m] \right) \mid b_1[m] \right\} &= b_1[m] \Re \left( w_1^H s_1 \right) = b_1[m] (w_1^H s_1), \tag{125} \\
\text{Var} \left\{ \Re \left( w_1^H r[m] \right) \mid b_1[m] \right\} &= \sum_{k=2}^{K} \left[ \Re \left( w_1^H s_k \right) \right]^2 + \eta \|w_1\|^2 + E \left\{ \left[ \Re \left( \Delta w_1^H r[m] \right) \right]^2 \right\}, \tag{126} \\
\text{where in (125) we used the fact that } w_1^H s_1 \text{ is real; and in (126) we have} \\
E \left\{ \left[ \Re \left( \Delta w_1^H r[m] \right) \right]^2 \right\} &= E \left\{ (\Re \Delta w_1^H \Re r[m] + \Im \Delta w_1^H \Im r[m])^2 \right\}
\end{align*}
\[ E \left\{ \mathbb{E}[w_1] \right\} = \mathbb{E}(b_1[m]) (w_1^H s_1), \]  

(128)

\[ \text{Var} \left\{ \mathbb{E}[w_1] \right\} = \sum_{k=2}^{K} \mathbb{E}(b_1[m])^2 \mathbb{E}(b_k[m])^2 + \frac{\eta}{2} \mathbb{E}[w_1] \]  

(129)

\[ E \left\{ |\Delta w_1|^2 \right\} = \text{tr} \left\{ E \left\{ |\Delta w_1|^2 \right\} C_r \right\} \geq \text{tr} (K_r C_r). \]  

(130)

**Appendix D: Proof of Proposition 7**

Define \( T_{\eta,\mu,\nu}^{x,y}(i,k,m,n) = \int_{CN} h_{\eta,\mu,\nu}(r;b^m,b^p) \, dr \) where \( h_{\eta,\mu,\nu}(r;b^m,b^p) \) is defined in (111)-(114).

Then for \( \theta_u = h_{i,j} \) and \( \theta_v = h_{k,l} \), we have

\[ \eta J_{u,v}^{x,y} = \frac{2}{4K} \sum_p \sum_q \sum_{m=1}^{N} \sum_{n=1}^{N} \left\{ \mathbb{E}[\mathcal{E}_j]_{n,i} \mathbb{E}[\mathcal{E}_j]_{n,k} \mathbb{E}[\mathcal{E}_j]_{n,j} \mathbb{E}[\mathcal{E}_j]_{n,l} \right\} T_{\eta,\mu,\nu}^{x,y}(m,n,p,q) \]

(131)

Using \( \lim_{\eta \to 0} T_{\eta,\mu,\nu}^{x,y}(i,k,m,n) = \delta_{i,k} \delta_{x,y} \delta_{m,n} \delta_{u,v} \), we have the following limits

\[ \lim_{\eta \to 0} \eta J_{u,v}^{x,y} = 2 \delta_{i,l} \sum_{n=1}^{N} \mathbb{E}[\mathcal{E}_j]_{n,i} \mathbb{E}[\mathcal{E}_j]_{n,k} \mathbb{E}[\mathcal{E}_j]_{n,j} \mathbb{E}[\mathcal{E}_j]_{n,l} = 2 \delta_{i,l} \mathbb{E}[\mathcal{E}_j]_{i,k} \]  

(132)
\[
\lim_{\eta \to 0} \eta J_{u,v}^{r,i} = 2\delta_{i,l} \sum_{n=1}^{N} \Re \{ [\mathbf{Z}_j]_{n,i} \} \Re \{ [\mathbf{Z}_j]_{n,k} \} + \Im \{ [\mathbf{Z}_j]_{n,i} \} \Im \{ [\mathbf{Z}_j]_{n,k} \} = 2\delta_{j,l} \Re \{ [\mathbf{Z}_j]_{i,l} \} \Im \{ [\mathbf{Z}_j]_{i,k} \} 
\]

(133)

\[
\lim_{\eta \to 0} \eta J_{u,v}^{r,v} = 0 \quad (134)
\]

References


