Asymptotic Analysis of Blind Multiuser Detection with Blind Channel Estimation

Anders Høst-Madsen *  Xiaodong Wang †  Seungjae Bahng ‡

Abstract
The analytical performance of the subspace-based blind linear MMSE multiuser detection algorithm in general multipath multi-antenna CDMA systems is investigated. In blind multiuser detection, the linear MMSE detector of a given user is estimated from the received signals, based on the knowledge of only the spreading sequence of that user. Typically, the channel of that user must be estimated first based on the orthogonality between the signal and noise subspaces. An asymptotic limit theorem for the estimate of the blind linear detector (when the received signal sample size is large) is obtained, based on which approximate expressions of the average output SINR’s and BER’s for both BPSK and QPSK modulations are given. Corresponding results for group-blind multiuser detectors are also obtained. Examples are provided to demonstrate the excellent match between the theory developed in this paper and the simulation results.

Index Terms:  Blind multiuser detection, blind channel estimation, group-blind, multipath, multi-antenna, signal subspace, asymptotic analysis.

1 Introduction

Considerable recent research has addressed the problem of subspace multiuser detection in CDMA systems [2, 7, 8, 14]. By and large, the research in this area has been focused on the development of signal processing algorithms to achieve improved receiver performance. And the performance assessment is largely done via computer simulations. The main difficulty in obtaining the analytical performance stems from the fact that in these adaptive methods, the detectors are estimated from the received signals; and those estimates coincide with the true detectors only when the number of received signals becomes infinitely large. First-order perturbation analysis on some subspace-based estimation methods was addressed in [1, 7, 11, 12]. Second-order perturbation analysis on

*Dept. of Electrical Engineering, University of Hawaii, Honolulu, HI 96822.
†Dept. of Electrical Engineering, Columbia University, New York, NY 10027.
‡Dept. of Electrical Engineering, University of Hawaii, Honolulu, HI 96822.
subspace methods was treated in [4, 6], and recently in [15]. In [5], an analytical framework has been developed for assessing the performance of several blind and group-blind multiuser detection methods; and the analytical performance developed there matches very well with the simulation results.

The system considered in [5] is a simple real-valued synchronous CDMA system with BPSK modulation signaling through AWGN channels. In this paper, we investigate the performance of blind multiuser detection in general multipath multi-antenna CDMA systems with arbitrary modulation. Typically in such an environment, the channel of the desired user must be estimated first, by exploiting the orthogonality between the signal and noise subspaces. A few recent works address issues that are related to those considered in this paper. In [3] the performance of a semi-blind channel estimation method, which linearly combines the least-squares fit on the training sequence and the blind subspace criterion, is analyzed. In [15, 16], perturbation analysis of the subspace blind channel estimation method is given. Our approach in assessing the performance of blind multiuser detection algorithms is to first establish the asymptotic distribution of the estimated blind detector, based on which performance indices such as output SINR and BER can then be obtained.

The rest of this paper is organized as follows. In Section 2, several systems under consideration are described, and the notion of a general complex Gaussian random vector is introduced. In Section 3, assuming the desired user’s channel is known, the performance of two blind multiuser detection algorithms, namely, the direct-matrix-inversion (DMI) method and the subspace method, is analyzed. In Section 4, the performance analysis of the subspace blind multiuser detection algorithm with blind channel estimation is given. In Section 5, simulation results are provided. Section 6 contains the conclusions. Moreover, some mathematical tools that are used in the analysis are collected in Appendix A. The proofs and derivations of the results in this paper are given in Appendices B, C and D.

2 Background

2.1 Synchronous Models

Synchronous Multipath CDMA System

We start by considering a $K$-user discrete-time synchronous multipath CDMA system with no ISI. The received $N$-dimensional signal during the $i$-th symbol interval in such a system can be
In blind multiuser detection, it is assumed that the receiver has only the knowledge of the spreading code and phase ambiguity if and only if \( \text{rank}(Q) = L \). Let the spreading waveform (with zero-padding when a guard interval is inserted) and the complex channel fading gain corresponding to the \( L \)-th path of the \( k \)-th user; \( n[i] \sim \mathcal{N}_c(0, \eta \mathbf{I}_N) \) is the circularly symmetric complex white Gaussian noise vector. In (1) we used the following notations: 

\[
S_k \triangleq [s_{1,k}, s_{2,k}, \ldots, s_{L,k}]; \quad h_k \triangleq [h_{1,k}, h_{2,k}, \ldots, h_{L,k}]^T; \quad \tilde{s}_k \triangleq S_k h_k; \quad \tilde{S} \triangleq [\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_K]; \quad \text{and } b[i] \triangleq \begin{bmatrix} b_1[i], b_2[i], \ldots, b_K[i] \end{bmatrix}^T.
\]

Let the autocorrelation matrix of the received signal \( r[i] \) and its eigendecomposition be

\[
C_r \triangleq E \{ r[i] r[i]^H \} = \tilde{S} \tilde{S}^H + \eta \mathbf{I}_N = U_s \Lambda_s U_s^H + \eta U_n U_n^H,
\]

where \( \Lambda_s \) is an diagonal matrix containing the largest \( K \) eigenvalues of \( C_r \); \( U_s \) contains the eigenvectors corresponding to the eigenvalues in \( \Lambda_s \); and \( U_n \) contains the \((N - K)\) eigenvectors corresponding to the smallest eigenvalue \( \eta \) of \( C_r \). Suppose that user 1 is the user of interest. Since \( \tilde{s}_1 \triangleq S_1 h_1 \), and \( U_n^H \tilde{s}_1 = 0 \), it then follows that \( h_1 \) can be obtained from the following relationship

\[
h_1 = \arg \min_{\|h\|=1} \|U_n^H (S_1 h)\|^2 = \arg \min_{\|h\|=1} \left| h^H \left( S_1^H U_n U_n^H S_1 \right) h \right| = \text{min eigenvector of } Q.
\]

Note that, however, (3) specifies \( h_1 \) only up to a scale and phase ambiguity, i.e., if \( h_1 \) is the solution to (3), so is \((\alpha e^{j\phi})h_1\) for any \( \alpha > 0 \) and \( \phi \). It is clear that \( h_1 \) is uniquely determined by (3) up to a scale and phase ambiguity if and only if \( \text{rank}(Q) = L - 1 \).

The (exact) linear MMSE detector for user 1 given by \([14]\)

\[
w_1 = C_r^{-1} \tilde{s}_1 = U_s \Lambda_s^{-1} U_s^H S_1 h_1.
\]

In blind multiuser detection, it is assumed that the receiver has only the knowledge of the spreading sequence of the desired user, \( S_1 \), and the detector is estimated from the received signals as follows. First the sample autocorrelation of the received signals is formed, and its eigendecomposition is computed:

\[
\hat{C}_r \triangleq \frac{1}{M} \sum_{i=0}^{M-1} r[i] r[i]^H = \hat{U}_s \hat{\Lambda}_s \hat{U}_s^H + \hat{U}_n \hat{\Lambda}_n \hat{U}_n^H.
\]

Then a channel estimation is performed by replacing the quantities in (3) by their estimates,

\[
h_1 = \text{min eigenvector of } \hat{Q}, \quad \text{with } \hat{Q} = S_1^H \hat{U}_n \hat{U}_n^H S_1.
\]
Finally, the estimated blind linear MMSE detector is given by
\[
\hat{w}_1 = \hat{U}_s \hat{A}_s^{-1} \hat{U}^H_s \hat{S}_1 \hat{h}_1.
\] (7)

**Synchronous Multi-antenna CDMA System**

We next consider a \(K\)-user synchronous CDMA system employing \(P\) receive antennas. Let \(h_{p,k}\) be the complex fading gain between the transmit antenna and the \(p\)-th receive antenna for the \(k\)-th user. At the \(p\)-th receive antenna, the received discrete-time signal during the \(i\)-th symbol interval is given by
\[
\mathbf{r}^{(p)}[i] = \sum_{k=1}^{K} h_{p,k} b_k[i] \mathbf{s}_k + \mathbf{n}^{(p)}[i], \quad p = 1, 2, \ldots, P,
\] (8)
where \(\mathbf{s}_k\) is the spreading waveform of the \(k\)-th user; \(\mathbf{n}^{(p)}[i] \sim \mathcal{N}_c(0, \eta I_N)\) is the circularly symmetric complex white Gaussian noise vector at antenna \(p\). It is assumed that the noise vectors at different antennas are independent. Denote \(\mathbf{h}_k \triangleq [h_{1,k}, h_{2,k}, \ldots, h_{P,k}]^T\), \(\mathbf{s}_k \triangleq h_k \otimes \mathbf{s}_k\), \(\mathbf{r}[i] \triangleq \left[\mathbf{r}^{(1)}[i]^T, \mathbf{r}^{(2)}[i]^T, \ldots, \mathbf{r}^{(P)}[i]^T\right]^T\), \(\mathbf{n}[i] \triangleq \left[\mathbf{n}^{(1)}[i]^T, \mathbf{n}^{(2)}[i]^T, \ldots, \mathbf{n}^{(P)}[i]^T\right]^T\), \(\mathbf{b}[i] \triangleq [b_1[i], b_2[i], \ldots, b_K[i]]^T\), and \(\mathbf{S} \triangleq [\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_K]\). Then (8) can be written in the same form as (1) with \(\mathbf{n}[i] \sim \mathcal{N}_c(0, \eta I_{PN})\). Moreover, the exact linear MMSE detector for user 1 is given by (4). And the subspace blind detector is obtained by following (5) and (6), with \(\hat{\mathbf{Q}} \triangleq (\mathbf{I}_P \otimes \mathbf{s}_1^H) \hat{U}_n \hat{U}^H_n (\mathbf{I}_P \otimes \mathbf{s}_1)\).

### 2.2 Asynchronous Models

We now consider a general asynchronous CDMA system with \(K\) users signaling through their respective multipath channels and employing \(P\) receive antennas. We start with the continuous-time signal model. Let the channel impulse response between the \(k\)-th user’s transmitter and the \(p\)-th receive antenna be
\[
g_k^{(p)}(t) = \sum_{l=1}^{L} \alpha_{l,k}^{(p)} \delta\left(t - \tau_{l,k}^{(p)}\right),
\] (9)
where \(L\) is the total number of paths; \(\alpha_{l,k}^{(p)}\) and \(\tau_{l,k}^{(p)}\) are respectively the complex path gain and the delay of the \(k\)-th user’s \(l\)-th path corresponding to the \(p\)-th receive antenna, \(\tau_{1,k}^{(p)} < \tau_{2,k}^{(p)} < \cdots < \tau_{L,k}^{(p)}\). The received continuous-time signal at the \(p\)-th receive antenna is given by
\[
\mathbf{r}^{(p)}(t) = \sum_{k=1}^{K} \sum_{i=0}^{M-1} b_k[i] \left\{ s_k(t - iT) * g_k^{(p)}(t) \right\} + \mathbf{n}^{(p)}(t)
\]
\[
= \sum_{k=1}^{K} \sum_{i=0}^{M-1} b_k[i] \sum_{l=1}^{L} \alpha_{l,k}^{(p)} s_k \left(t - iT - \tau_{l,k}^{(p)}\right) + \mathbf{n}^{(p)}(t),
\] (10)
where $\ast$ denotes convolution and $s_k(t)$ is the spreading waveform of the $k$-user given by

$$s_k(t) = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} s_{j,k} \psi(t - jT_c), \quad 0 \leq t < T,$$

(11)

where $N$ is the processing gain; $\{ s_{j,k} \}_{j=0}^{N-1}$ is a signature sequence of $\pm1$'s assigned to the $k$-th user; and $\psi(\cdot)$ is a chip waveform of duration $T_c = \frac{T}{N}$ and with unit energy, i.e., $\int_0^{T_c} \psi(t)^2 \mathrm{d}t = 1.$

At the receiver, the received signal $r^{(p)}(t)$ is filtered by a chip-matched filter and sampled at the chip-rate. Let

$$t \overset{\triangle}{=} \max_{1 \leq k \leq K, \ 1 \leq p \leq P} \left\{ \left\| \frac{\tau^{(p)}_{L,k} + T_c}{T} \right\| \right\},$$

be the maximum delay spread in terms of symbol intervals. Substituting (11) into (10), the $q$-th signal sample during the $i$-th symbol is given by

$$r^{(p)}_q[i] = \int_{iT + q T_c}^{iT + (q+1) T_c} r^{(p)}(t) \psi(t - iT - qT_c) \mathrm{d}t$$

$$= \int_{iT + q T_c}^{iT + (q+1) T_c} \psi(t - iT - qT_c) \sum_{k=1}^{K} \sum_{m=0}^{M-1} b[k][m] \sum_{l=1}^{L} \alpha^{(p)}_{l,k} \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} s_{j,k} \psi(t - mT - \tau^{(p)}_{l,k} - jT_c) \mathrm{d}t + n^{(p)}_q[i],$$

(12)

$$= \sum_{k=1}^{K} \sum_{m=0}^{M-1} b[k][m] \sum_{j=0}^{N-1} s_{j,k} \sum_{l=1}^{L} \frac{1}{\sqrt{N}} \int_{iT + q T_c}^{iT + (q+1) T_c} \psi(t) \psi(t - mT - \tau^{(p)}_{l,k} - jT_c) \mathrm{d}t + n^{(p)}_q[i],$$

$$= \sum_{k=1}^{K} \sum_{m=0}^{M-1} b[k][m] \sum_{j=0}^{N-1} s_{j,k} \sum_{l=1}^{L} \frac{1}{\sqrt{N}} \int_{iT + q T_c}^{iT + (q+1) T_c} \psi(t) \psi(t - mT - \tau^{(p)}_{l,k} - jT_c) \mathrm{d}t + n^{(p)}_q[i],$$

$$q = 0, \ldots, N-1; \ i = 0, \ldots, M-1.$$

where $n^{(p)}_q[i] = \int_{iT + q T_c}^{iT + (q+1) T_c} n^{(p)}(t) \psi(t - iT - qT_c) \mathrm{d}t.$ Denote

$$r^{(p)}_q[i] \overset{N \times 1}{=} \begin{bmatrix} r^{(p)}_0[i] \\ \vdots \\ r^{(p)}_{N-1}[i] \end{bmatrix}, \quad b[i] \overset{K \times 1}{=} \begin{bmatrix} b_1[i] \\ \vdots \\ b_K[i] \end{bmatrix}, \quad n^{(p)}_q[i] \overset{N \times 1}{=} \begin{bmatrix} n^{(p)}_0[i] \\ \vdots \\ n^{(p)}_{N-1}[i] \end{bmatrix},$$

$$s^{(p)}_j[i] \overset{N \times K}{=} \begin{bmatrix} s^{(p)}_1[jN] & \cdots & s^{(p)}_K[jN] \\ \vdots & \ddots & \vdots \\ s^{(p)}_1[jN + N-1] & \cdots & s^{(p)}_K[jN + N-1] \end{bmatrix}, \quad j = 0, \ldots, t.$$
Then (12) can be written in terms of vector convolution as

\[ r^{(p)}[i] = \tilde{S}^{(p)}[i] * b[i] + n^{(p)}[i], \quad p = 1, \ldots, P. \]  

(13)

By stacking \( m \) successive sample vectors, we define the following quantities

\[
\begin{align*}
\begin{bmatrix}
\begin{array}{c}
\{ \hat{r}^{(p)}[i] \} \\
\vdots \\
\hat{r}^{(p)}[i + m - 1]
\end{array}
\end{bmatrix}, \\
\begin{bmatrix}
\begin{array}{c}
\{ \hat{n}^{(p)}[i] \} \\
\vdots \\
\hat{n}^{(p)}[i + m - 1]
\end{array}
\end{bmatrix}, \\
\begin{bmatrix}
\begin{array}{c}
\{ \hat{b}[i] \} \\
\vdots \\
\hat{b}[i + m - 1]
\end{array}
\end{bmatrix}
\end{align*}
\]

\[
\begin{bmatrix}
\begin{array}{c}
\{ \tilde{s}^{(p)}[i] \} \\
\vdots \\
\tilde{s}^{(p)}[i + m - 1]
\end{array}
\end{bmatrix}
\]

We can then write (13) in a matrix forms as

\[ r^{(p)}[i] = \tilde{S}^{(p)} b[i] + n^{(p)}[i], \quad p = 1, \ldots, P. \]  

(14)

Finally denote

\[
\begin{align*}
\begin{bmatrix}
\begin{array}{c}
\{ r^{(1)}[i] \} \\
\vdots \\
\hat{r}^{(P)}[i]
\end{array}
\end{bmatrix}, \\
\begin{bmatrix}
\begin{array}{c}
\{ s^{(1)}[i] \} \\
\vdots \\
\hat{s}^{(P)}[i]
\end{array}
\end{bmatrix}, \\
\begin{bmatrix}
\begin{array}{c}
\{ n^{(1)}[i] \} \\
\vdots \\
\hat{n}^{(P)}[i]
\end{array}
\end{bmatrix}
\end{align*}
\]

Then (14) can be written as

\[ r[i] = \tilde{S} b[i] + n[i]. \]  

(15)

The smoothing factor \( m \) is chosen according to

\[ m = \left\lceil \frac{NP + K}{NP - K} \right\rceil \iota. \]

Note that for such \( m \), the matrix \( \tilde{S} \) is a “tall” matrix, i.e., \( PNm \geq K(m + \iota) \).

Let the autocorrelation matrix of the augmented received signal and its eigendecomposition be

\[ C_r \triangleq E \{ r[i] r[i]^H \} = \tilde{S} \tilde{S}^H + \eta I_{NPm} = U_s A_s U_s^H + \eta U_n U_n^H, \]  

(16)

where we assume that the matrix \( \tilde{S} \) has full column rank, and hence the signal subspace has a dimension \( K(m + \iota) \). Denote \( \tilde{s}_1 \) as the \( (K\iota + 1) \)-th column of \( \tilde{S} \). The linear MMSE detector for detecting the \( i \)-th bit of user 1, \( b_1[i] \), based on \( r[i] \) in (15) is then given by

\[ w_1 = C_r^{-1} \tilde{s}_1 = U_s A_s^{-1} U_s^H \tilde{s}_1. \]  

(17)
We next address the method for estimating $\tilde{s}_1$. From (12),

$$\tilde{s}_k[p] = \sum_{j=0}^{N-1} s_{j,k} h_k[p][n-j], \quad n = 0, 1, \ldots, (\ell+1)N - 1,$$

(18)

with $h_k[p][n] \triangleq \frac{1}{\sqrt{N}} \sum_{l=1}^{L} \alpha_{l,k}^{(p)} \int_{0}^{T_c} \psi(t) \psi\left(t - \tau_{l,k}^{(p)} + nT_c\right), \quad n = 0, 1, \ldots, \mu - 1,$

(19)

where the length of the channel response $\{h_k[p][n]\}_{n=0}^{\mu-1}$ satisfies

$$\mu = \left[\frac{\tau_{L,k}^{(p)} T_c}{T}\right] \leq \ell N.$$

(20)

Denote

$$\tilde{s}_k[p] = \left[\begin{array}{c}
\tilde{s}_k^{(p)}[0] \\
\vdots \\
\tilde{s}_k^{(p)}[(\ell+1)N - 1]
\end{array}\right]_{(\ell+1)N \times 1}, \quad h_k[p] = \left[\begin{array}{c}
h_k^{(p)}[0] \\
\vdots \\
h_k^{(p)}[(\ell N - 1)]
\end{array}\right]_{\mu \times 1},$$

and

$$\Xi_k = \left[\begin{array}{cccc}
s_{0,k} & s_{1,k} & \cdots & s_{0,k} \\
s_{1,k} & s_{0,k} & \cdots & \vdots \\
\vdots & \vdots & \ddots & \vdots \\
s_{N-1,k} & \cdots & \cdots & s_{0,k} \\
s_{N-1,k} & \vdots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots \\
s_{N-1,k} & \cdots & \cdots & s_{N-1,k}
\end{array}\right]_{(\ell+1)N \times \mu}.$$

Then (18) can be written in matrix form as

$$\tilde{s}_k[p] = \Xi_k h_k[p].$$

(21)

Finally, let $\bar{s}_k$ be the $(K\ell + k)$-th column of the matrix $\bar{S}$. Then we have

$$\bar{s}_k = \left[\begin{array}{c}
\tilde{s}_k^{(1)}[1] \\
\vdots \\
\tilde{s}_k^{(P)}[1] \\
0_{(m-\ell-1)PN \times 1}
\end{array}\right]_{PNm \times 1} = \left[\begin{array}{c}
\Xi_k \\
\vdots \\
\Xi_k \\
0_{(m-\ell-1)PN \times P\mu}
\end{array}\right]_{PNm \times P\mu} \left[\begin{array}{c}
h_k^{(1)}[1] \\
\vdots \\
h_k^{(P)}[1] \\
h_k[1]
\end{array}\right]_{P\mu \times 1}.$$
In order to estimate the desired user’s channel \( h_1 \), we again resort to the orthogonality condition

\[
\begin{align*}
  h_1 &= \arg \min_{\|h\|=1} \| U_n^H \hat{\Xi}_1 h \|^2 \\
&= \arg \min_{\|h\|=1} h^H \left( \hat{\Xi}_1^H U_n U_n^H \hat{\Xi}_1 \right) h = \text{min-eig-vec}(Q).
\end{align*}
\]

(23)

Note that \( h_1 \) is determined uniquely by (23) up to a scale and phase ambiguity if and only if \( \text{rank}(Q) = P - 1. \) A blind estimate of the linear MMSE detector for user 1 is then given by the following procedure.

\[
\begin{align*}
  C_r &\triangleq \frac{1}{M} \sum_{i=1}^{M} r[i] r[i]^H = \hat{U}_s \hat{\Lambda}_s \hat{U}_s^H + \hat{U}_n \hat{\Lambda}_n \hat{U}_n^H, \\
  \hat{Q} &= \hat{\Xi}_1^T \hat{U}_n \hat{U}_n^H \hat{\Xi}_1, \\
  \hat{h}_1 &= \text{min-eig-vec}(\hat{Q}), \\
  \hat{w}_1 &= \hat{U}_s \hat{\Lambda}_s^{-1} \hat{U}_s^H \hat{\Xi}_1 \hat{h}_1.
\end{align*}
\]

(24) (25) (26) (27)

**Remark:** From the above discussion, it is seen that the three systems considered here share the same signal model of the form

\[
r[i] = \tilde{S} b[i] + n[i].
\]

(28)

Furthermore, for these three systems, a linear MMSE decision on the \( i \)-th symbol of user 1 is made based on the output of the linear MMSE detector \( w_1^H r[i] \), where \( w_1 \) has the form

\[
w_1 = C_r^{-1} \hat{s}_1 = U_s \Lambda_s^{-1} U_s^H \hat{s}_1.
\]

(29)

Moreover, the composite signature waveform \( \hat{s}_1 \) of the desired user is determined by the original signature waveform of this user, and a channel vector \( h_1 \), which is given by the minimum eigenvector of a matrix \( Q \). The matrix \( Q \) is in turn determined by the noise subspace \( U_n \) and the original signature waveform of the desired user.

In the remaining of this paper, when analyzing the performance of the blind receivers, we focus on the synchronous multipath CDMA system described in Section 2.1. The results, however, are directly applicable to the synchronous multi-antenna CDMA system described in Section 2.1, and the asynchronous multipath multi-antenna CDMA system described in Section 2.2, with appropriate interpretation of the quantities such as \( \Lambda_s, U_s, U_n, \) and \( Q \) in the corresponding system.

### 2.3 Complex Multivariate Gaussian Distribution

Let \( \mathbf{x} \in \mathbb{C}^n \) be a complex random vector. We say that \( \mathbf{x} \) is complex Gaussian distributed if the vector \( [\Re \mathbf{x}^T, \Im \mathbf{x}^T]^T \in \mathbb{R}^{2n} \) has a (real-valued) Gaussian distribution. Hence a complex Gaussian vector
$x$ is completely specified by its mean $\mu \triangleq E\{x\}$, and the covariance matrix $\text{Cov}\left\{ [\Re x^T, \Im x^T]^T \right\}$. An equivalent characterization of the complex Gaussian vector $x$ is through the following two complex-valued covariance matrices

$$C \triangleq E\{ (x - \mu)(x - \mu)^H \}, \quad \text{and} \quad \tilde{C} \triangleq E\{ (x - \mu)(x - \mu)^T \}.$$  

We call $C$ the Hermitian covariance matrix and $\tilde{C}$ the symmetric covariance matrix. The real-valued covariance matrix can be expressed by $C$ and $\tilde{C}$ as follows:

$$\text{Cov}\left\{ \begin{bmatrix} \Re x \\ \Im x \end{bmatrix} \right\} = \begin{bmatrix} \text{Cov}\{\Re x, \Re x\} & \text{Cov}\{\Re x, \Im x\} \\ \text{Cov}\{\Im x, \Re x\} & \text{Cov}\{\Im x, \Im x\} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \Re C + \Re \tilde{C} & \Im C - \Im \tilde{C} \\ \Im \tilde{C} + \Im C & \Re \tilde{C} - \Re C \end{bmatrix}. \quad (30)$$

Hence in what follows, we use the notation $x \sim \mathcal{N}_c(\mu, C, \tilde{C})$ to represent a complex Gaussian vector. When $\tilde{C} = 0$, $x$ is said to have a circularly symmetric complex Gaussian distribution. In this case, $\Re x$ and $\Im x$ are independent and have the same (real-valued) Gaussian distribution.

### 3 Performance of Blind MUD with Known Channels

In this section, we assume that the receiver has the knowledge of original signature waveform and the channel of the desired user, i.e., $S_1$ and $h_1$ in the synchronous multipath CDMA case, $s_1$ and $h_1$ in the synchronous multi-antenna CDMA case, and $\Xi_1$ and $h_1$ in the asynchronous multipath multi-antenna CDMA case. Equivalently, the composite signature waveform $\tilde{s}_1$ is assumed known. This corresponds to systems where the desired user’s channel is obtained through non-blind methods, such as pilot channels or pilot symbols.

As mentioned earlier, we focus on the synchronous multipath CDMA system described in Section 2.1.1. Based on the sample autocorrelation matrix $C_r$ in and its eigendecomposition in (5), we can obtain two forms of the estimated linear MMSE detector, namely, the direct-matrix-inversion (DMI) detector, and the subspace detector, given respectively by

$$\hat{w}_1 = \tilde{C}_r^{-1}\tilde{s}_1, \quad \text{DMI blind detector} \quad (31)$$

and

$$\hat{w}_1 = \tilde{U}_s\tilde{A}_s^{-1}\tilde{U}_s^H\tilde{s}_1, \quad \text{subspace blind detector} \quad (32)$$

In what follows, we first present results on the asymptotic distributions of the DMI blind detector (31) and the subspace blind detector (32), when the number of received signals $M$ is large. We then give expressions of the output signal-to-interference-plus-noise ratio (SINR) as well as the bit error rate (BER) for the two blind detectors.
3.1 Asymptotic Distributions of Blind Detectors

The following result gives the asymptotic distribution of the DMI blind detector and that of the subspace blind detector. It is a generalization of a similar result in [5] (which is for real-valued signals and binary antipodal modulation) to complex-valued signals and arbitrary modulation scheme. The proof is given in Appendix B.

**Theorem 1** Let \( \mathbf{w}_1 = \mathbf{C}_r^{-1} \hat{\mathbf{s}}_1 = \mathbf{U}_s \mathbf{A}_s^{-1} \mathbf{U}_s^H \hat{\mathbf{s}}_1 \) be the exact linear MMSE detector, and let \( \hat{\mathbf{w}}_1 \) be the DMI blind detector given by (31) or the subspace blind detector given by (32). Then

\[
\sqrt{M} (\hat{\mathbf{w}}_1 - \mathbf{w}_1) \to N_c \left( \mathbf{0}, \mathbf{C}_w^0, \mathbf{C}_w^0 \right), \quad \text{in distribution, as } M \to \infty,
\]

with

\[
\mathbf{C}_w^0 = (\mathbf{w}_1^H \hat{\mathbf{s}}_1) \mathbf{U}_s \mathbf{A}_s^{-1} \mathbf{U}_s^H + \tau \mathbf{U}_n \mathbf{U}_n^H
+ \mathbf{U}_s \mathbf{A}_s^{-1} \mathbf{U}_s^H \left[ \mu \left( \hat{\mathbf{S}}_1^T \hat{\mathbf{w}}_1 \hat{\mathbf{S}}_1^* \right) + \nu \mathbf{D} \right] \hat{\mathbf{S}}_1^H \mathbf{U}_s \mathbf{A}_s^{-1} \mathbf{U}_s^H,
\]

\[
\tilde{\mathbf{C}}_w^0 = \mathbf{w}_1 \mathbf{w}_1^T + \mathbf{U}_s \mathbf{A}_s^{-1} \mathbf{U}_s^H \left[ \mu \left( \mathbf{w}_1^T \hat{\mathbf{S}}_1^* \hat{\mathbf{S}}_1 \mathbf{w}_1 \right) \mathbf{I}_K + \nu \mathbf{D} \right] \hat{\mathbf{S}}_1^T \mathbf{U}_s \mathbf{A}_s^{-1} \mathbf{U}_s^T,
\]

where

\( \mathbf{D} \triangleq \text{diag} \left\{ |\hat{\mathbf{s}}_1^H \mathbf{w}_1|^2, |\hat{\mathbf{s}}_2^H \mathbf{w}_1|^2, \ldots, |\hat{\mathbf{s}}_K^H \mathbf{w}_1|^2 \right\} \),

\( \tilde{\mathbf{D}} \triangleq \text{diag} \left\{ (\hat{\mathbf{s}}_1^H \mathbf{w}_1)^2, (\hat{\mathbf{s}}_2^H \mathbf{w}_1)^2, \ldots, (\hat{\mathbf{s}}_K^H \mathbf{w}_1)^2 \right\} \),

\( \mu \triangleq \left| E \{ |\mathbf{b}|^2 \} \right|^2, \)

\( \nu \triangleq E \left\{ |\mathbf{b}|^4 \right\} - 2 E \left\{ |\mathbf{b}|^2 \right\}^2 - E \{ |\mathbf{b}|^2 \}^2, \)

\( \tau \triangleq \begin{cases} \frac{1}{\eta} \hat{\mathbf{s}}_1^H \mathbf{U}_s \mathbf{A}_s^{-1} \mathbf{U}_s^H \hat{\mathbf{s}}_1, & \text{DMI blind detector} \\ \eta \hat{\mathbf{s}}_1^H \mathbf{U}_s \mathbf{A}_s^{-1} (\mathbf{A}_s - \eta \mathbf{I}_K)^{-2} \mathbf{U}_s^H \hat{\mathbf{s}}_1, & \text{subspace blind detector} \end{cases} \).

and where \( \mathbf{b} \) denotes the transmitted modulation symbols.

In particular, for BPSK modulation, \( \mathbf{b} \in \{+1, -1\} \). Then \( \mu = 1 \) and \( \nu = -2 \). And for QPSK modulation, \( \mathbf{b} \in \left\{ \frac{1+j}{\sqrt{2}}, \frac{1-j}{\sqrt{2}}, \frac{-1+j}{\sqrt{2}}, \frac{-1-j}{\sqrt{2}} \right\} \). Hence \( \mu = 0 \) and \( \nu = -1 \).

3.2 Bit Error Rate for BPSK and QPSK

For BPSK signaling \( \mathbf{b}_k[i] \in \{+1, -1\} \), the decision rule is given by \( \hat{\mathbf{b}}_1[i] = \text{sign} \left( \Re \left\{ \hat{\mathbf{w}}_1^H \mathbf{r}[i] \right\} \right) \). Denote the decision statistic \( \xi \triangleq \Re \left\{ \hat{\mathbf{w}}_1^H \mathbf{r}[i] \right\} \), then we have

\[
\xi = \Re \left\{ (\mathbf{w}_1 + \Delta \mathbf{w}_1)^H \left( \hat{\mathbf{s}}_1 \mathbf{b}_1[i] + \sum_{k=2}^{K} \hat{\mathbf{s}}_k \mathbf{b}_k[i] + \mathbf{n}[i] \right) \right\}
= \Re \left\{ \mathbf{w}_1^H \hat{\mathbf{s}}_1 \right\} \mathbf{b}_1[i] + \sum_{k=2}^{K} \Re \left\{ \mathbf{w}_1^H \hat{\mathbf{s}}_k \right\} \mathbf{b}_k[i] + \Re \left\{ \mathbf{w}_1^H \mathbf{n}[i] \right\}
\]
The first term of (40) contains the data bit of the desired user. The third term is Gaussian. For fixed $b_2[i], \ldots, b_K[i]$, the fourth and the fifth terms are asymptotically Gaussian by Theorem 1. The last term is not Gaussian, but following [5] it can be shown that it is asymptotically Gaussian when $N, M \to \infty$ under some technical conditions. Hence conditioned on $b_2[i], \ldots, b_K[i]$, $\xi$ can be argued to be asymptotically Gaussian, i.e., for finite $N$ and $M$ a conditional Gaussian distribution for $\xi$ is a good approximation. Moreover, as is the case with the exact linear MMSE detector [9], the unconditional distribution of $\xi$ turns out to be well approximated by Gaussian. This means that the BER can be well approximated by using a single $Q$-function. We have the following mean and variance

$$E\{\xi | b_1[i]\} = b_1[i] \cdot w_1^H s_1,$$

$$\text{Var}\{\xi | b_1[i]\} = \sum_{k=2}^K E\{\Delta w_1^H s_k\}^2 + \frac{\eta}{2} \|w_1\|^2 + E\{\|\Delta w_1^H r[i]\|^2\}, \quad (41)$$

where in (41) we used the fact that $w_1^H s_1$ is real; and in (42) we have

$$E\{[\Re (\Delta w_1^H r[i])]^2\} = \frac{1}{4} E\{ (\Delta w_1^H r[i] + r[i]^T \Delta w_1)^2 \}$$

$$= \frac{1}{4} E\{ (\Delta w_1^H r[i] + r[i]^H \Delta w_1) (r[i]^T \Delta w_1^* + \Delta w_1^T r[i]^*) \}$$

$$= \frac{1}{4M} \text{tr} \left[ E\{\Delta w_1 \Delta w_1^T\}^* E\{r[i]r[i]^T\} + 2E\{\Delta w_1 \Delta w_1^T\} E\{r[i]r[i]^H\} + E\{\Delta w_1 \Delta w_1^T\} E\{r[i]r[i]^T\}^* \right]$$

$$= \frac{1}{4M} \text{tr} \left( C_w^0 C_r + C_w^0 C_r^* \right) + \frac{1}{2M} \text{tr} \left( C_w^0 C_r \right) = \frac{1}{2M} \text{tr} \left[ E\{\Delta w_1 \Delta w_1^T\} \right], \quad (43)$$

where $C_r$ and $C_w^0$ are given respectively by (2) and (33), $C_w^0 \triangleq E\{\Delta w \Delta w^T\}$ is given by (34), and since the noise vector $n[i]$ is circularly symmetric complex Gaussian, we have

$$C_r \triangleq E\{r[i]r[i]^T\} = \tilde{S} \tilde{S}^T. \quad (44)$$

Hence under the Gaussian approximation of $\xi$, the BER of the blind detector under BPSK modulation is then given by

$$P_b(e) \cong Q(\sqrt{\gamma}), \quad (45)$$

with

$$\gamma = \frac{(w_1^H s_1)^2}{\sum_{k=2}^K [\Re (w_1^H s_k)]^2 + \frac{\eta}{2} \|w_1\|^2 + \frac{1}{2M} \text{tr} \left[ E\{\Delta w_1 \Delta w_1^T\} \right]}. \quad (46)$$
On the other hand, for QPSK signaling, \( b_k[i] \in \left\{ \frac{1+j}{\sqrt{2}}, \frac{1-j}{\sqrt{2}}, -\frac{1+j}{\sqrt{2}}, -\frac{1-j}{\sqrt{2}} \right\} \), the decision rule for the two bits associated with the symbol \( b_1[i] \) is given by

\[
\hat{b}_1[i] = \text{sign} \left( \Re \left\{ \hat{w}_1^H r[i] \right\} \right), \quad \text{and} \quad \hat{b}_1[i] = \text{sign} \left( \Im \left\{ \hat{w}_1^H r[i] \right\} \right). \tag{47}
\]

Denote \( \xi^r \triangleq \Re \left\{ \hat{w}_1^H r[i] \right\} \) and \( \xi^i \triangleq \Im \left\{ \hat{w}_1^H r[i] \right\} \). Similarly as above, we approximate \( \xi^r \) and \( \xi^i \) as Gaussian distributed, with the mean and the variance given respectively by

\[
E\{\xi^r | b_1[i]\} = b_1[i] \cdot w_1^H s_1, \quad \text{and} \quad E\{\xi^i | b_1[i]\} = b_1[i] \cdot w_1^H s_1, \tag{48}
\]

\[
\text{Var}\{\xi^r | b_1[i]\} = \text{Var}\{\xi^i | b_1[i]\}
= \sum_{k=2}^K \left[ \Re \left( w_1^H \hat{s}_k \right)^2 E\{b_k^2[i]\} + \Im \left( w_1^H \hat{s}_k \right)^2 E\{b_k^2[i]\} \right] + \frac{\eta}{2} \| w_1 \|^2 + E \left\{ \Re \left( \Delta w_1^H r[i] \right) \right\}^2 \tag{49}
= \frac{1}{2} \sum_{k=2}^K \left| w_1^H \hat{s}_k \right|^2 + \frac{\eta}{2} \| w_1 \|^2 + \frac{1}{2M} \text{tr} \left( C_0^0 C_r \right). \tag{50}
\]

The last term of (49) is given by (43), except that for QPSK modulation we have \( C_r = 0 \). Moreover, straightforward calculation shows that \( \text{Cov}\{\xi^r, \xi^i | b_1[i]\} = 0 \). Hence, under the Gaussian approximation of \( \xi^r \) and \( \xi^i \), the BER of the blind detector under QPSK modulation is given by

\[
P_b(e) \approx Q(\sqrt{\gamma}), \tag{51}
\]

with

\[
\gamma = \frac{(w_1^H s_1)^2 \sum_{k=2}^K \left| w_1^H \hat{s}_k \right|^2 + \eta \| w_1 \|^2 + \frac{1}{M} \text{tr} \left( C_0^0 C_r \right)}{\sum_{k=2}^K \left| w_1^H \hat{s}_k \right|^2 + \eta \| w_1 \|^2 + \frac{1}{M} \text{tr} \left( C_0^0 C_r \right)}. \tag{52}
\]

4 Performance of Blind MUD with Blind Channel Estimation

4.1 Asymptotic Distribution of Blind Detector

The following result gives the asymptotic distribution of the blind detector with blind channel estimation, given by (5)-(6). The proof is given in Appendix C.

**Theorem 2** Let \( w_1 = U_s A_s^{-1} U_s^H \hat{s}_1 \) be the exact linear MMSE detector, \( h_1 \) be the true channel of user 1, and \( \tilde{w}_1 \) be the estimated blind detector given by (5)-(6). Let \( C_0^0 \) and \( C_w^0 \) be the quantities given by (33) and (34) respectively. Then there exists a phase factor \( e^{j\phi} \) such that

\[
\sqrt{M} \left( \tilde{w}_1 - \| h_1 \|^{-1} e^{j\phi} w_1 \right) \to N_c(0, \| h_1 \|^{-2} C_w, \| h_1 \|^{-2} C_w), \quad \text{in distribution, as } M \to \infty,
\]
with \( C_w = C_w^0 + \beta_1 \Psi Q^\dagger \Psi^H + \beta_2 \left( \Psi Q^\dagger S_1^H U_n U_n^H + U_n U_n^H S_1 Q^\dagger \Psi^H \right) \), \( (53) \)

\[ \bar{C_w} = \bar{C_w}^0, \quad (54) \]

where \( \Psi \triangleq U_s A_s^{-1} U_s^H S_1 \), \( (55) \)

\[ \beta_1 \triangleq \eta \tilde{s}_1^H U_s A_s (A_s - \eta I_K)^{-2} U_s^H \tilde{s}_1, \quad (56) \]

\[ \beta_2 \triangleq \eta \tilde{s}_1^H U_s (A_s - \eta I_K)^{-2} U_s^H \tilde{s}_1, \quad (57) \]

and \( Q^\dagger \) is the pseudo-inverse of the matrix \( Q \triangleq S_1^H U_n U_n^H S_1 \).

We can also obtain the asymptotic distribution of the channel estimate, which appeared in [16], as follows.

**Corollary 1** Let \( h_1 \) be the true channel of user 1, and let \( \hat{h}_1 \) be the channel estimate given by (6). Then there exists a phase factor \( e^{j\phi} \), such that

\[ \sqrt{M} \left( \hat{h}_1 - \|h_1\|^{-1} e^{j\phi} h_1 \right) \xrightarrow{\text{in distribution}} \mathcal{N}_c \left( 0, \beta_1 \|h_1\|^{-2} Q^\dagger, 0 \right), \quad \text{as } M \to \infty, \]

where \( \beta_1 \) is given by (56). Thus asymptotically \( \hat{h}_1 \) is circularly symmetric complex Gaussian.

Hence the blind detector with blind channel estimation can be written as

\[ \hat{w}_1 = \|h_1\|^{-1} e^{j\phi} w_1 + \Delta w_1. \quad (58) \]

Using (53) and after some manipulations, we obtain

\[ \text{tr}(C_w C_r) = \text{tr}(C_w^0 C_r) + \beta_1 \text{tr} \left[ \left( S_1^H U_s A_s^{-1} U_s^H S_1 \right) Q^\dagger \Gamma \right], \quad (59) \]

where \( C_w^0 \) is given by (33) and

\[ \beta_1 = \eta \left[ I_K + \eta \left( \tilde{S}_1^H \tilde{S}_1 \right)^{-1} \right]_{1,1}. \quad (60) \]

The derivations of (59) and (60) are given in Appendix D. Hence it is clear that the effect of channel estimation is the additional term \( \beta_1 \text{tr}(\Gamma) \) in (59).

### 4.2 Bit Error Rate for BPSK and QPSK

As seen in (58), the blind detector \( \hat{w}_1 \) has an amplitude and phase ambiguity. The amplitude ambiguity does not affect the detector performance, but the phase ambiguity must be estimated. Suppose that an estimate of the phase ambiguity \( e^{j\phi} \) is available. Then for BPSK modulation, the
decision rule becomes \( \hat{b}_1[i] = \text{sign} \left( \Re \left\{ e^{-j\phi} \hat{w}_1^H r[i] \right\} \right) \). As before, under the Gaussian approximation of the decision statistic, the bit error rate is given by

\[
P_b(e) \cong Q(\sqrt{\gamma}),
\]

with
\[
\gamma = \frac{(w_1^H \tilde{s}_1)^2}{\sum_{k=2}^{K} [\Re (w_1^H \tilde{s}_k)]^2 + \frac{\eta}{2} \|w_1\|^2 + \frac{1}{2M} \text{tr} \left[ \Re \left( C_0^w C_r \right) + C_0^w C_r + \beta_1 \Gamma \right]}.
\]

On the other hand, for QPSK modulation, the decision rule is given by

\[
\hat{b}_1^r[i] = \text{sign} \left( \Re \left\{ e^{-j\phi} \hat{w}_1^H r[i] \right\} \right), \quad \text{and} \quad \hat{b}_1^i[i] = \text{sign} \left( \Im \left\{ e^{-j\phi} \hat{w}_1^H r[i] \right\} \right).
\]

The bit error rate in this case is given by

\[
P_b(e) \cong Q(\sqrt{\gamma}),
\]

with
\[
\gamma = \frac{(w_1^H \tilde{s}_1)^2}{\sum_{k=2}^{K} |w_1^H \tilde{s}_k|^2 + \eta \|w_1\|^2 + \frac{1}{M} \text{tr} \left( C_0^w C_r + \beta_1 \Gamma \right)}.
\]

5 Extension to Group-Blind MUD

In some scenarios, such as the uplink of a cellular system, the receiver has knowledge of more than one user’s spreading code. In [13] a number of detectors utilizing this knowledge were derived, called group-blind detectors. Assume that the receiver has knowledge of \( \tilde{K} \) users’ spreading codes, i.e., \( S_1, \ldots, S_{\tilde{K}} \). For the known channel case, the channels for these users, \( h_1, \ldots, h_{\tilde{K}} \) are also assumed known; whereas for the unknown channels case, the receiver estimates the channel of each known user, \( h_1, \ldots, h_{\tilde{K}} \), using the blind method in (3). Denote by \( \tilde{S} \) the matrix consisting of the composite codes of the \( \tilde{K} \) known users, i.e., the first \( \tilde{K} \) columns of \( \tilde{S} \) in (1).

Several group-blind detectors were derived in [13], but we will here concentrate on the hybrid form-II since this is the most suitable one when combined with blind channel estimation. The detector is given by

\[
w_1 = U_s \Lambda_s^{-1} U_s^H \tilde{S} \left( \tilde{S}^H U_s \Lambda_s^{-1} U_s^H \tilde{S} \right)^{-1} \tilde{e}_1,
\]

where \( \tilde{e}_k \) is a vector of all zeros except for the \( k \)-th entry, which is one. We now have the following result regarding the asymptotic distribution of the estimated group-blind detector, with and without channel estimation. Its proof is quite tedious and follows similar steps as in the proofs of Theorems 1 and 2; and will therefore be omitted here.
Theorem 3 Let \( \mathbf{w}_1 = \mathbf{U}_s \mathbf{A}_s^{-1} \mathbf{U}_s^H \tilde{\mathbf{S}} \left( \tilde{\mathbf{S}}^H \mathbf{U}_s \mathbf{A}_s^{-1} \mathbf{U}_s^H \tilde{\mathbf{S}} \right)^{-1} \tilde{\mathbf{e}}_1 \) be the exact group-blind detector, and let \( \hat{\mathbf{w}}_1 \) be the estimated one. Then, for known channels

\[
\sqrt{M} (\hat{\mathbf{w}}_1 - \mathbf{w}_1) \to \mathcal{N}_c \left( 0, \mathbf{C}_w^0, \mathbf{C}_w^0 \right), \quad \text{in distribution as } M \to \infty,
\]

with

\[
\mathbf{C}_w^0 = \mathbf{Q} (\mathbf{w}_1^H \mathbf{v}_1) \mathbf{U}_s \mathbf{A}_s^{-1} \mathbf{U}_s^H \mathbf{Q}^H + \tau \mathbf{U}_n \mathbf{U}_n^H - \mathbf{Q} \mathbf{U}_s \mathbf{A}_s^{-1} \mathbf{U}_s^H \mathbf{S} \mathbf{D} \mathbf{S}^H \mathbf{U}_s \mathbf{A}_s^{-1} \mathbf{U}_s^H \mathbf{Q}^H, \quad (67)
\]

where

\[
\begin{align*}
\mathbf{v}_1 & \triangleq \tilde{\mathbf{S}} (\tilde{\mathbf{S}}^H \mathbf{U}_s \mathbf{A}_s^{-1} \mathbf{U}_s^H \tilde{\mathbf{S}})^{-1} \tilde{\mathbf{e}}_1, \\
\mathbf{Q} & \triangleq \mathbf{I}_N - \mathbf{U}_s \mathbf{A}_s^{-1} \mathbf{U}_s^H \tilde{\mathbf{S}} (\tilde{\mathbf{S}}^H \mathbf{U}_s \mathbf{A}_s^{-1} \mathbf{U}_s^H \tilde{\mathbf{S}})^{-1} \tilde{\mathbf{S}}^H, \\
\mathbf{D} & \triangleq \text{diag} \left\{ |s_1^H \mathbf{w}_1|^2, |s_2^H \mathbf{w}_1|^2, \ldots, |s_K^H \mathbf{w}_1|^2 \right\}, \\
\tilde{\mathbf{D}} & \triangleq \text{diag} \left\{ (s_1^H \mathbf{w}_1)^2, (s_2^H \mathbf{w}_1)^2, \ldots, (s_K^H \mathbf{w}_1)^2 \right\}, \\
\text{and } \tau & \triangleq \eta \mathbf{w}_1^H \mathbf{U}_s \mathbf{A}_s^{-1} (\mathbf{A}_s - \mathbf{I}_K)^{-2} \mathbf{U}_s^H \mathbf{v}_1. \quad (73)
\end{align*}
\]

For blindly estimated channels the Hermitian covariance matrix\(^1\) of \( \hat{\mathbf{w}}_1 \) is given by

\[
\mathbf{C}_w = \mathbf{C}_w^0 + \mathbf{P}_1 \left( \mathbf{U} \mathbf{M}_1 \mathbf{Y}^H + \mathbf{Y} \mathbf{M}_1^H \mathbf{U}^H + \mathbf{P}_1 \mathbf{Y} \mathbf{M}_2 \mathbf{Y}^H \right) \mathbf{P}_1^H + \mathbf{P}_2 \mathbf{M}_3 \mathbf{P}_2^H, \quad (74)
\]

where

\[
\begin{align*}
\mathbf{Y} & \triangleq \mathbf{U}_s \mathbf{A}_s^{-1} \mathbf{U}_s^H, \\
\mathbf{v}_2 & \triangleq (\tilde{\mathbf{S}}^H \mathbf{Y} \tilde{\mathbf{S}})^{-1} \tilde{\mathbf{e}}_1, \\
\mathbf{P}_1 & \triangleq \mathbf{I}_N - \mathbf{Y} \tilde{\mathbf{S}} (\tilde{\mathbf{S}}^H \mathbf{Y} \tilde{\mathbf{S}})^{-1} \tilde{\mathbf{S}}^H, \\
\mathbf{P}_2 & \triangleq \mathbf{Y} \tilde{\mathbf{S}} (\tilde{\mathbf{S}}^H \mathbf{Y} \tilde{\mathbf{S}})^{-1}, \quad (78)
\end{align*}
\]

and \( \mathbf{Q}_{k,s}^\dagger \) is the matrix \( \mathbf{Q}^\dagger \) defined in (144) but for the \( k \)-th user. The matrices \( \mathbf{M}_1, \mathbf{M}_2, \) and \( \mathbf{M}_3 \) are defined by

\[
\begin{align*}
[M_1]_{ij} &= \sum_{k=1}^K [\mathbf{v}_{2,k}]^* [0 \cdots 0 \cdots 0] \gamma_k \cdots \gamma_k |U^H \mathbf{S}_k| Q_{k,s}^H |S_{j,s}|^H, \\
\text{with } \gamma_k &= \eta \sum_{n=1}^K \frac{1}{(\lambda_n - \eta)^2 |U^H \tilde{s}_k|^2}, \quad (79)
\end{align*}
\]

\[
[M_2]_{ij} = \sum_{k=1}^K \sum_{l=1}^K [\mathbf{v}_{2,l}]^* |S_{j,s}^\dagger (U^H \mathbf{S}_k)^H M_{kl} U^H \mathbf{S}_k Q_{l,s}^H |S_{j,s}^\dagger |^2 [\mathbf{v}_{2,l}]^*, \quad (80)
\]

\[
[M_3]_{ij} = \mathbf{v}_1^H Y^H S_{j,s} Q_{j,s}^\dagger (U^H S_{j})^H M_{j,s} U^H S_{j} Q_{j,s}^\dagger S_{j,s}^\dagger Y \mathbf{v}_1, \quad (81)
\]

---

\(^1\)We will only list the Hermitian covariance matrix, as this is all that is needed for SINR and BER expressions for QPSK modulation.
where \( M_{kl} \) is a diagonal matrix with elements
\[
[M_{kl}]_{ij} = \begin{cases} 
\eta \sum_{n=1}^{K} \frac{\lambda_n}{(\lambda_n - \eta)^2} [U^H s_k]_n[U^H \tilde{s}_l]_n^*, & i = j \text{ and } i, j > K, \\
0, & \text{otherwise.}
\end{cases}
\]

As before, using the above expressions for \( C_w \) we can then obtain the SINR and BER expressions for the group-blind detector.

6 Simulations

6.1 Asymptotic Distribution of Blind Detectors

In what follows we first verify through simulation Theorems 1 and 2 on the asymptotic covariances of the blind detectors. In order to verify Theorem 1, for a fixed signal block size \( M \), we compute many realizations of the estimated blind detector \( \hat{w}_1 \) and the corresponding error \( \Delta w_1 = \hat{w}_1 - w_1 \).

Let \( \hat{E}(\cdot) \) denote the sample mean operator and \( \|\cdot\|_F \) denote the Frobenius norm of a matrix. Define
\[
\epsilon_{\Delta} \overset{\Delta}{=} \left\| \hat{E}\left\{ \Delta w_1 \Delta w_1^H \right\} - \frac{1}{M} C_w^0 \right\|_F, \qquad \bar{\epsilon}_{\Delta} \overset{\Delta}{=} \frac{\epsilon_{\Delta}}{\|1/\sqrt{M}C_w^0\|_F},
\]
\[
\bar{\epsilon}_{\Delta} \overset{\Delta}{=} \left\| \hat{E}\left\{ \Delta w_1 \Delta w_1^T \right\} - \frac{1}{M} \bar{C}_w^0 \right\|_F, \quad \bar{\epsilon}_{\Delta} \overset{\Delta}{=} \frac{\bar{\epsilon}_{\Delta}}{\|1/\sqrt{M}C_w^0\|_F}.
\]

We can then plot \( \epsilon_{\Delta} \) and \( \bar{\epsilon}_{\Delta} \) to indicate how well the simulation results match with Theorem 1.

On the other hand, the verification of Theorem 2 is more subtle, because it involves an unknown phase ambiguity term \( e^{j\phi} \), which prevents a direct comparison between simulation and analytical results (like the one discussed above), as also observed for a similar case in [4]. To overcome this difficulty, we need to first normalize the phases of both the estimated detector and the exact detector. The following lemma, which also appeared [16], states in general how phase transformation influences the asymptotic covariances. Note that [4] also treats the similar situation but in a somewhat different way then our approach below. We use \( [x]_k \) to denote the \( k \)-th element of vector \( x \), \([C]_{i,j}\) to denote the \((i, j)\)-th element of matrix \( C \), and \([C]_{i,k}\) to denote the \( k \)-th column of matrix \( C \).

Lemma 1 Suppose that \( \hat{x} \) is a \( C^1 \) estimator of \( x \) based on \( M \) signal samples, and that
\[
\sqrt{M} \left( \hat{x} - e^{j\phi} x \right) \to \mathcal{N}_c(0, C_x, \Sigma_x), \quad \text{in distribution, as } M \to \infty,
\]
for some phase factor \( e^{j\phi} \). Then
\[
\sqrt{M} \left( \hat{x}[x]_1^* - x[x]_1^* \right) \to \mathcal{N}_c(0, \Sigma_x, \Sigma_x), \quad \text{in distribution, as } M \to \infty,
\]
where

\[
\Sigma_x = |x_1|^2 C_x + |C_x|_1 x x^H + |x_1|_1 x [C_x]_{1,1}^H + |x_1|_1^* [C_x]_{1,1} x^H, \quad (85)
\]

\[
\bar{\Sigma}_x = (|x_1|^2)^* C_x + |C_x|_1 x x^T + |x_1|_1 x [C_x]_{1,1}^T + |x_1|_1^* [C_x]_{1,1} x^T. \quad (86)
\]

**Proof:** Since \( \hat{x} \) is \( C^1 \) it has a differential. Let \( \Delta x \) be the differential of \( \hat{x} \) at \( e^{j\phi} x \). Then the differential of \( \hat{x} [x_1]^* \) at \( e^{j\phi} x \) is given by

\[
\Delta (\hat{x} [x_1]^*) = x [\Delta x]^* + \Delta x [x_1]^* . \quad (87)
\]

By Lemma 3, we have \( \Sigma_x = \lim_{M \to \infty} M \cdot E \{ \Delta (\hat{x} [x_1]^*) \Delta (\hat{x} [x_1]^*)^H \} \), and \( \bar{\Sigma}_x = \lim_{M \to \infty} M \cdot E \{ \Delta (\hat{x} [x_1]^*) \Delta (\hat{x} [x_1]^*)^T \} \). Substituting (87) into these, we obtain (85) and (86).

Hence based on the above lemma, we verify Theorem 2 in the following way. For a fixed signal block size \( M \), we compute many realizations of the estimated blind detector \( \hat{w}_1 \) and define the following error

\[
\Delta \hat{w}_1 \triangleq [\hat{w}_1]^* \hat{w}_1 - [w_1]^* w_1 . \quad (88)
\]

Define

\[
\epsilon \triangleq \left\| \frac{1}{M} \Sigma_w \right\|_F, \quad \bar{\epsilon} \triangleq \left\| \frac{1}{M} \bar{\Sigma}_w \right\|_F . \quad (89)
\]

where \( \Sigma_w \) and \( \bar{\Sigma}_w \) are transformations of \( C_w \) and \( \bar{C}_w \) respectively similar to (85) and (86). We can then plot \( \epsilon \) and \( \bar{\epsilon} \) to indicate how well the simulation results match with Theorem 2.

The simulated system is a five-user (\( K = 5 \)) synchronous CDMA system. Each user’s original spreading sequence is randomly generated and has length 5. The channel of each user has length 4 and is randomly generated. A guard interval of 3 chips long between two consecutive symbols to avoid intersymbol interference. Each user employs BPSK modulation. In Figure 1 we plot the relative error \( \epsilon \) and \( \bar{\epsilon} \) versus the signal block size \( M \) for the subspace blind detector for the cases of both known and unknown channel. In Figure 2, we illustrate the convergence behavior by plotting the diagonal elements of the exact and estimated Hermitian covariance matrix \( C_w \), and the absolute values of the diagonal elements of the exact and estimated symmetric covariance matrix \( \bar{C}_w \). According to Theorems 1 and 2 \( \lim_{M \to \infty} M \bar{C}_w = C_w \) and \( \lim_{M \to \infty} M \bar{C}_w = C_w \), and the figure illustrates this.
6.2 SINR and BER in Synchronous Multipath CDMA Systems

We next compare the SINR and BER expressions of the blind detectors in Sections 3 and 4 with simulation results. The simulated system is a synchronous multipath CDMA system described in Section 2.1.1. The number of users is $K = 18$. Each user’s original spreading sequence is randomly generated and has length 21. The channel of each user has length 11 and is randomly generated. Both the spreading sequences and the channels of all users are fixed throughout the simulations. Note that the length of the composite signature waveform of each user (i.e., $\tilde{s}_k$) is 31. Here we insert a guard interval of length 10 chips between two consecutive symbols to avoid intersymbol interference. Note such a setup is merely for the purpose of verifying the theoretical results in this paper.

As noted earlier, when the channel is unknown and is blindly estimated, the blind channel estimator introduces a phase ambiguity $\alpha = e^{j\phi}$ which must be estimated for detection purpose. In the simulations, we employ the following simple phase estimator:

$$\hat{\alpha} = \left[ \frac{1}{M} \sum_{i=1}^{M} (\hat{w}_i^H r[i])^2 \right]^{\frac{1}{2}} = \left[ \hat{w}_1^H \tilde{C}_r \hat{w}_1^* \right]^{\frac{1}{2}}. \quad (91)$$

Note that the above phase estimator still contains a phase ambiguity of $\pi$ for BPSK and $\pi/2$ for QPSK, which is inherent to any blind detector.
Figure 2: Illustration of the convergence of the covariance matrix estimates. Left: The circles represent the theoretical values of the diagonal elements of $\hat{C}_w$, and the lines represent the corresponding estimated values. It is seen that $\lim_{M \to \infty} M \hat{C}_w = C_w$. Right: The circles represent the theoretical absolute values of the diagonal elements of $\bar{C}_w$, and the lines represent the corresponding estimated values.
In Figure 3, assuming BPSK modulation the simulated output SINR of three blind detectors, namely, the DMI detector with known channel, the subspace detector with known channel, and the subspace detector with unknown channel, are plotted as a function of input $E_b/N_0$ of each user, for two block sizes, i.e., $M = 300$ and $M = 2000$. The corresponding theoretical values [given by (46) for known channel and (62) for unknown channel], as well as the theoretical and simulated SINR of the exact MMSE detector, are also plotted in the same figure. The corresponding simulated and theoretical BER curves are given in Figure 4. [The theoretical BER is given by (45) for known channel and by (61) for unknown channel]. Moreover, the SINR and BER results for QPSK modulation for the same simulated system are plotted in Figures 5 and 6. [The theoretical performance under QPSK is given by (51)-(52) for known channel, and (64)-(65) for unknown channel.] It is seen from these figures that both the SINR and the BER expressions obtained in this paper match very well with the simulation results. Further more, in unknown channels, the simple phase estimator (91) incurs little performance loss compared with the case where the phase ambiguity is perfectly known. In addition, the output SINR as a function of the block size $M$ are plotted for BPSK and QPSK in Figures 7 and 8 respectively.

### 6.3 Results for Synchronous Multi-antenna CDMA System

Next we give some simulation results for the synchronous multi-antenna CDMA system described in Section 2.1.2. The simulated system is synchronous CDMA system with spreading gain $N = 5$, number of antennas $L = 5$, and number of users $K = 32$. The user spreading sequences and channels are randomly generated and fixed throughout the simulations. QPSK modulation is assumed. The simulated and theoretical BER performance of different detectors are plotted in Figure 9. Again the excellent match between the theory developed in this paper and the simulation results is evident.

### 6.4 Results for Asynchronous Multipath CDMA with ISI

Finally we present results for an asynchronous multipath CDMA system with ISI, for both blind and group-blind detectors. The system has 11 users (8 known, 3 unknown) with a spreading gain of 31, and the randomly generated channel has a delay spread of 11 chips. The signal processing and detector construction for this case is described in Section 2.2. Specifically, the smoothing factor is set as $m = 3$. The simulated and analytical SINR and BER results for both the blind and group-blind detectors are plotted in Figures 10–13. The close match between the analytical expressions and the simulation results indicate the theories developed in this paper are very useful in assessing the performance of blind and group-blind detectors under sophisticated channel conditions.
Figure 3: Output SINR versus input $E_b/N_0$ for BPSK.

Figure 4: BER versus input $E_b/N_0$ for BPSK.
Figure 5: Output SINR versus input $E_b/N_0$ for QPSK.

Figure 6: BER versus input $E_b/N_0$ for QPSK.
Figure 7: Output SINR versus signal block size $M$ for BPSK.

Figure 8: Output SINR versus signal block size $M$ for QPSK.
Figure 9: BER versus $E_b/N_0$ for a synchronous multi-antenna CDMA system with QPSK modulation.

Figure 10: Output SINR versus signal block size $M$ for QPSK in asynchronous multipath CDMA with ISI, blind detector.
Figure 11: BER versus signal block size $M$ for QPSK in asynchronous multipath CDMA with ISI, blind detector.

Figure 12: Output SINR versus signal block size $M$ for QPSK in asynchronous multipath CDMA with ISI, group-blind detector.
Figure 13: BER versus signal block size $M$ for QPSK in asynchronous multipath CDMA with ISI, group-blind detector.

7 Conclusions

We have provided an analytical framework for assessing the performance of the subspace blind multiuser detector in CDMA systems with multipath and multiple antennas. The estimation of the blind multiuser detector involves computing the signal subspace components from the received signals, and a blind channel estimation based on the orthogonality between the signal and noise subspaces and the knowledge of the signature sequence of the given user. We have established an asymptotic limit theorem for the estimated blind detector (when the number of received signals is large). We have also obtained expressions of the output SINR and BER of the blind detector under BPSK and QPSK modulations. It is seen that the analytical performance obtained in this paper matches very well with the simulation results. Finally we have extended our analysis to group-blind multiuser detection and obtained the corresponding asymptotic performance results, which again closely match with simulations.

Appendix A: Some Useful Lemmas

In this appendix, we provide some lemmas which are useful for proving the main results in this paper. A complex-valued random matrix is said to be Gaussian distributed, if the joint distribution
of all its elements is complex Gaussian. First we establish that the sample autocorrelation matrix \( \hat{C}_r \) is asymptotically complex Gaussian distributed as the sample size \( M \to \infty \). This is the generalization of Lemma 2 in [5] to complex-valued covariance matrices and general modulation schemes.

**Lemma 2** Define \( \Delta C_r \triangleq \hat{C}_r - C_r \), where \( C_r \) and \( \hat{C}_r \) are give by (2) and (5) respectively. Then \( \sqrt{M} \Delta C_r \) converges in probability towards a complex Gaussian matrix with mean \( 0 \) and \( (N^2 \times N^2) \) Hermitian and symmetric covariance matrices whose elements are specified by

\[
M \cdot \text{Cov} \left\{ [\Delta C_r]_{i,j}, [\Delta C_r]_{m,n} \right\} = |C_r|_{i,m} |C_r|_{j,n} + \mu \left[ \bar{S}S^T \right]_{i,m} \left[ \bar{S}S^T \right]_{j,n} + \nu \sum_{\alpha=1}^{K} |s_\alpha| [\bar{s}_\alpha] [\bar{s}_\alpha] [\bar{s}_\alpha] [s_\alpha] n, \quad (92)
\]

\[
M \cdot \text{Cov} \left\{ [\Delta C_r]_{i,j}, [\Delta C_r]_{m,n} \right\} = |C_r|_{i,m} |C_r|_{j,m} + \mu \left[ \bar{S}S^T \right]_{i,m} \left[ \bar{S}S^T \right]_{j,m} + \nu \sum_{\alpha=1}^{K} |s_\alpha| [\bar{s}_\alpha] [\bar{s}_\alpha] [s_\alpha] n, \quad (93)
\]

where \( \mu = E \{ b^2 \} \), and \( \nu = E \{ |b|^4 \} - 2E \{ |b|^2 \}^2 - E \{ b^2 \}^2 \).

**Proof:** Since \( \hat{C}_r \) given by (5) has \( E \{ \hat{C}_r \} = C_r \), and it is a sum of i.i.d. terms \( (r[i]r[i]^T) \), by Theorem 1.9.1B in [10], it is asymptotically Gaussian as a real \( 2N^2 \)-dimensional vector. The Hermitian and symmetric \( (N^2 \times N^2) \) covariance matrices are given by the covariances of the zero-mean complex random matrix \( (r[i]r[i]^H) \). To calculate this covariance, note that (for notational convenience, in what follows we drop the time index \( i \).)

\[
[r r]^H_{i,j} = \sum_{\alpha=1}^{K} \sum_{\beta=1}^{K} |s_\alpha| [\bar{s}_\beta] [s_\beta] [\bar{s}_\beta] m |s_\lambda] n Cov \{ b_\alpha b_\beta, b_\beta b_\lambda \} + \sum_{\alpha=1}^{K} \sum_{\beta=1}^{K} |s_\alpha| [\bar{s}_\beta] [s_\beta] [\bar{s}_\beta] m Cov \{ b_\alpha n_\beta, b_\beta n_\lambda \} + \sum_{\alpha=1}^{K} \sum_{\beta=1}^{K} |s_\alpha| [\bar{s}_\beta] [s_\beta] [\bar{s}_\beta] m Cov \{ b_\alpha n_\beta, b_\beta n_\lambda \} + \sum_{\alpha=1}^{K} \sum_{\beta=1}^{K} |s_\alpha| [\bar{s}_\beta] [s_\beta] [\bar{s}_\beta] m Cov \{ b_\alpha n_\beta, b_\beta n_\lambda \} (95)
\]

For the Hermitian covariance matrix we now have

\[
\text{Cov} \left\{ [r r]^H_{i,j}, [r r]^H_{m,n} \right\} = \sum_{\alpha=1}^{K} \sum_{\beta=1}^{K} |s_\alpha| [\bar{s}_\beta] [s_\beta] [\bar{s}_\beta] m Cov \{ b_\alpha n_\beta, b_\beta n_\lambda \} + \sum_{\alpha=1}^{K} \sum_{\beta=1}^{K} |s_\alpha| [\bar{s}_\beta] [s_\beta] [\bar{s}_\beta] m Cov \{ b_\alpha n_\beta, b_\beta n_\lambda \} + \sum_{\alpha=1}^{K} \sum_{\beta=1}^{K} |s_\alpha| [\bar{s}_\beta] [s_\beta] [\bar{s}_\beta] m Cov \{ b_\alpha n_\beta, b_\beta n_\lambda \} + \sum_{\alpha=1}^{K} \sum_{\beta=1}^{K} |s_\alpha| [\bar{s}_\beta] [s_\beta] [\bar{s}_\beta] m Cov \{ b_\alpha n_\beta, b_\beta n_\lambda \}. (96)
\]

Here

\[
\text{Cov} \{ b_\alpha b_\beta, b_\beta b_\lambda \} = \delta_{\alpha=\gamma} \delta_{\beta=\gamma} E \{ |b|^2 \} + \delta_{\alpha=\gamma} \delta_{\beta=\gamma} E \{ |b|^2 \}^2 + \delta_{\alpha=\beta=\gamma} \left( E \{ |b|^4 \} - 2E \{ |b|^2 \}^2 - E \{ b^2 \}^2 \right),
\]

\[
\text{Cov} \{ b_\alpha n_\beta, b_\beta n_\lambda \} = \delta_{\alpha=\beta} \delta_{\beta=\gamma} E \{ |b|^2 \},
\]

\[
\text{Cov} \{ b_\alpha n_\beta, b_\beta n_\lambda \} = \delta_{\alpha=\beta} \delta_{\beta=\gamma} E \{ |b|^2 \},
\]

\[
\text{Cov} \{ n_\alpha n_\beta, n_\beta n_\lambda \} = \delta_{\alpha=\beta} \delta_{\beta=\gamma} E \{ |b|^2 \}.
\]

27
Then
\[
\text{Cov}\left\{ \left[rr^H\right]_{i,j}, \left[rr^H\right]_{m,n}^* \right\} = \sum_{\alpha=1}^{K} \sum_{\beta=1}^{K} \left[ \tilde{s}_{\alpha,i}^* \tilde{s}_{\alpha,j} \tilde{s}_{\beta,j}^* \tilde{s}_{\beta,j} \right] \text{E}\{|b|^2\}^2 + \text{Cov}_{\alpha,i} \left[ \tilde{s}_{\alpha,n}^* \tilde{s}_{\alpha,j} \right] \text{E}\{|b|^2\}^2
\]
+ \eta E\{|b|^2\} \sum_{\alpha=1}^{K} \left[ \tilde{s}_{\alpha,i}^* \tilde{s}_{\alpha,j} \tilde{s}_{\alpha,j} \right] \text{E}\{|b|^2\}^2 + \left( E\{|b|^4\} - 2E\{|b|^2\}^2 - |E\{b^2\}|^2 \right).
\]
\[
\sum_{\alpha=1}^{K} \left[ \tilde{s}_{\alpha,i}^* \tilde{s}_{\alpha,j} \right] \text{E}\{|b|^2\}^2 + \left( E\{|b|^4\} - 2E\{|b|^2\}^2 - |E\{b^2\}|^2 \right) \sum_{\alpha=1}^{K} \left[ \tilde{s}_{\alpha,i}^* \tilde{s}_{\alpha,j} \right] \text{E}\{|b|^2\}^2 + \eta \delta_{i=j}.
\]

Note that (92) is the same as (97). To get (93), notice that since \( \Delta C_r \) is Hermitian, \( [\Delta C_r]_{m,n} = [\Delta C_r]_{n,m}^* \), so that
\[
\text{Cov}\{[\Delta C_r]_{i,j}, [\Delta C_r]_{m,n}\} = \text{Cov}\{[\Delta C_r]_{i,j}, [\Delta C_r]_{n,m}^*\}.
\]
Therefore (93) follows from (92).

An important tool used in [5] is the differential of a function of (real-valued) matrices. Here we need to make use of differential of a function of complex-valued matrices. Consider a function \( f : \mathbb{C}^n \rightarrow \mathbb{C}^m \). The differential of \( f \) at a point \( x_0 \) is a bilinear function \( L_f(\cdot, \cdot ; x_0) : \mathbb{C}^n \times \mathbb{C}^n \rightarrow \mathbb{C}^m \), such that
\[
\forall \varepsilon > 0, \exists \delta > 0 : \|x - x_0\| < \delta \Rightarrow \|f(x) - f(x_0) - L_f(x - x_0, x^* - x_0^*; x_0)\| < \varepsilon.
\]

If the differential exists, it is given by \( L_f(x; x_0) = T(x_0)x + \bar{T}(x_0)x^* \), where
\[
T(x_0) \triangleq \frac{\partial f}{\partial x}|_{x=x_0} \triangleq \frac{1}{2} \left[ \frac{\partial f}{\partial x} - J \frac{\partial f}{\partial x} \right]_{x=x_0},
\]
\[
\bar{T}(x_0) \triangleq \frac{\partial f}{\partial x^*}|_{x=x_0} \triangleq \frac{1}{2} \left[ \frac{\partial f}{\partial x} + J \frac{\partial f}{\partial x} \right]_{x=x_0},
\]
where the derivatives on the right-hand sides of (101) and (102) should be understood as real-valued derivatives. Note that \( f \) is holomorphic if and only if \( T(x_0) = 0 \). We now have the following generalization of Theorem 3.3A in [10] (Lemma 3 in [5]), which is related to the result in [16].
Lemma 3 Suppose that $x(M) \in \mathbb{C}^n$ is asymptotically complex Gaussian, i.e.,

$$\sqrt{M} [x(M) - x_0] \rightarrow N_c(0, C_\lambda, \overline{C_\lambda}), \text{ in distribution, as } M \rightarrow \infty.$$ Let $f : \mathbb{C}^n \rightarrow \mathbb{C}^m$ be a function. Denote $y(M) = f[x(M)]$. Suppose that $f$ has a nonzero differential $L_f(x, x^*; x_0) = T(x_0)x + \overline{T}(x_0)x^*$ at $x_0$. Denote $\Delta x(M) \triangleq x(M) - x_0$, and $\Delta y(M) = T(x_0)\Delta x(M) + \overline{T}(x_0)\Delta x^*(M)$. Then

$$\sqrt{M} [y(M) - f(x_0)] \rightarrow N_c(0, C_y, \overline{C_y}), \text{ in distribution, as } M \rightarrow \infty, \quad (103)$$

where $C_y = \lim_{M \rightarrow \infty} M \cdot E \left\{ \Delta y(M)\Delta y(M)^H \right\}$, \quad (104)

$$\overline{C_y} = \lim_{M \rightarrow \infty} M \cdot E \left\{ \Delta y(M)\Delta y(M)^T \right\}. \quad (105)$$

In what follows, we will make use of the following identities of matrix differentials.

$$C = f(X) \triangleq MX \implies \Delta C = M\Delta X, \quad (106)$$

$$C = f(X, Y) \triangleq XY \implies \Delta C = X\Delta Y + \Delta XY, \quad (107)$$

$$C = f(X) \triangleq X^{-1} \implies \Delta C = -X^{-1}\Delta XX^{-1}. \quad (108)$$

These functions are all holomorphic, so that the differentials do not depend on $\Delta X^*$.

The following is an extension of lemma 4 in [5] to complex-valued matrices.

Lemma 4 Let the $N \times N$ Hermitian matrix $C_0$ have an eigendecomposition $C_0 = U_0\Lambda_0U_0^H$, where the eigenvalues satisfy $\lambda_1^0 > \lambda_2^0 > \cdots > \lambda_K^0 > \lambda_{K+1}^0 = \lambda_{K+2}^0 = \cdots = \lambda_N^0$. Let $\Delta C$ be a Hermitian variation of $C_0$ and denote $C \triangleq C_0 + \Delta C$. Let $T$ be a unitary transformation of $C$ as

$$T(C) \triangleq U_0^HCU_0. \quad (109)$$

Denote the eigendecomposition of $T$ as

$$T = WAW^H. \quad (110)$$

where $W$ is constrained so that $[W]_{i,i} > 0$ for $i \leq K$. (Note that if $C = C_0$, then $W = I_N$ and $A = \Lambda_0$.) The differential of $A$ at $\Lambda_0$, and the differential of $W$ at $I_N$, as a function of $\Delta T = U_0\Delta CU_0^H$, are given respectively by

$$\Delta \lambda_k = [\Delta T]_{k,k}, \quad 1 \leq k \leq K, \quad (111)$$

$$[\Delta W]_{i,k} = \begin{cases} 0, & i = k, \\ \frac{1}{\lambda_i^0 - \lambda_k^0}[\Delta T]_{i,k}, & 1 \leq i \leq N, 1 \leq k \leq K. \end{cases} \quad (112)$$
Proof: We only show the lemma for the case when all eigenvalues are different. The general case follows by a continuity argument similar to the case of real-valued matrices [5]. Note that the constraint $[W]_{i,i} > 0$ is to ensure that the eigenvalue decomposition (110) is unique.

Consider the following function

$$f(X, Y, Z) = (X - YZY^H, I_N - Y^HY),$$

(113)

where $Z$ is a real diagonal matrix, $X$ is Hermitian and $Y$ is a complex matrix with real diagonal elements. Clearly, $f(C_0, W, A) = 0$ and $f$ is differentiable with respect to $(X, Y, Z)$ (as a real function). The differential of $f(T, W, A)$ at $(A_0, I_N, A_0)$ is given as follows

$$\Delta (T - WAW^H) = \Delta T - \Delta W\Lambda - \Delta \Lambda - \Lambda \Delta W^H,$$

(114)

$$\Delta (I_N - W^HW) = \Delta W + \Delta W^H.$$

(115)

By setting the above differentials to zeros and solving for $\Delta \Lambda$ and $\Delta W$ as a function of $\Delta T$, we obtain the lemma for the case when the eigenvalues are distinct. □

Appendix B: Proof of Theorem 1

DMI Blind Detector - Consider the function $\hat{C}_r \rightarrow \hat{w}_1 = \hat{C}_r^{-1} \hat{s}_1$. The differential of $\hat{w}_1$ at $C_r$ is given by

$$\Delta \hat{w}_1 = -C_r^{-1} \Delta C_r C_r^{-1} \hat{s}_1,$$

(116)

where $\Delta C_r \triangleq \hat{C}_r - C_r$. Then according to Lemma 3, $\sqrt{M} (\hat{w}_1 - w_1)$ is asymptotically complex Gaussian as $M \rightarrow \infty$, with zero-mean and covariance matrices given by

$$C_w \triangleq M \cdot E \{ \Delta w_1 \Delta w_1^H \} = M \cdot C_r^{-1} \{ \Delta C_r C_r^{-1} \hat{s}_1 \hat{s}_1^H C_r^{-1} \Delta C_r C_r^{-1} \}$$

(117)

$$\hat{C}_w \triangleq M \cdot E \{ \Delta w_1 \Delta w_1^T \} = M \cdot E \{ C_r^{-1} \Delta C_r C_r^{-1} \hat{s}_1 \hat{s}_1^T C_r^{-1} \Delta C_r C_r^{-1} \}$$

(118)

Now, by Lemma 2, we have

$$M \cdot E \{ \Delta C_r w_1 w_1^H \Delta C_r \}_{i,j}$$

$$= M \cdot E \left\{ \sum_{m=1}^{N} \sum_{n=1}^{N} [\Delta C_r]_{i,m} [w_1]_m [\Delta C_r]_{n,j} [w_1^*]_n \right\}$$

30
Substituting (120) into (117), we get

\[ \begin{align*}
&= \sum_{m=1}^{N} \sum_{n=1}^{N} (C^r_{i,j}[C^*_r]_{m,n} + \mu [\tilde{S}\tilde{S}^T]_{i,n} [\tilde{S}\tilde{S}^T]^*_{m,j} + \nu \sum_{\alpha=1}^{K} [\tilde{s}_\alpha]_i [\tilde{s}_\alpha^*]_n [\tilde{s}_\alpha]_j) [w_1]_m [w_1^*]_n \\
&= [C^r_{i,j}] \left( \sum_{m=1}^{N} \sum_{n=1}^{N} [C^*_r]_{m,n} [w_1]_m [w_1^*]_n \right) + \mu \left( \sum_{m=1}^{N} [\tilde{S}\tilde{S}^T]_{i,n} [w_1]_m \right) \left( \sum_{n=1}^{N} [\tilde{S}\tilde{S}^T]^*_{i,n} [w_1^*]_n \right) \\
&\quad + \nu \sum_{\alpha=1}^{K} [\tilde{s}_\alpha]_i [\tilde{s}_\alpha^*]_j \sum_{m=1}^{N} [\tilde{s}_\alpha]_m [w_1]_m \sum_{n=1}^{N} [\tilde{s}_\alpha]_n [w_1^*]_n.
\end{align*} \]

Writing (119) in matrix form, we have

\[ M \cdot E \{ \Delta C^r w_1^H \Delta C^r \} = (w^H_1 C^r w_1) C^r + \mu (\tilde{S}\tilde{S}^T) w^*_1 w^*_1 (\tilde{S}\tilde{S}^T)^* C^{-1}_r + \nu \tilde{S} \tilde{D} \tilde{S}^H C^{-1}_r, \quad (120) \]

with \( D \triangleq \text{diag} \{ |s^H_1 w_1|^2, |s^H_2 w_1|^2, \ldots, |s^H_K w_1|^2 \} \).

Substituting (120) into (117), we get

\[ C_w = (w^H_1 C^r w_1) C^{-1}_r + \mu C^{-1}_r (\tilde{S}\tilde{S}^T) w^*_1 w^*_1 (\tilde{S}\tilde{S}^T)^* C^{-1}_r + \nu \tilde{S} \tilde{D} \tilde{S}^H C^{-1}_r \]

(121)

Finally (33) follows from (121) by expressing \( C^r \) in terms of its eigendecomposition.

On the other hand, also by Lemma 2, we have

\[ M \cdot E \{ \Delta C^r w_1^H \Delta C^r \}_{i,j} = M \cdot E \left\{ \sum_{m=1}^{N} \sum_{n=1}^{N} \left( \sum_{\alpha=1}^{K} [\tilde{s}_\alpha]_i [\tilde{s}_\alpha^*]_j \sum_{m=1}^{N} [\tilde{s}_\alpha]_m [w_1]_m \sum_{n=1}^{N} [\tilde{s}_\alpha]_n [w_1^*]_n \right) [w_1]_m [w_1^*]_n \right\} \]

\[ = \sum_{m=1}^{N} \sum_{n=1}^{N} (C^r_{i,n}[C^*_r]_{m,j} + \mu [\tilde{S}\tilde{S}^T]_{i,j} [\tilde{S}\tilde{S}^T]^*_{m,n} + \nu \sum_{\alpha=1}^{K} [\tilde{s}_\alpha]_i [\tilde{s}_\alpha^*]_j \sum_{m=1}^{N} [\tilde{s}_\alpha]_m [w_1]_m \sum_{n=1}^{N} [\tilde{s}_\alpha]_n [w_1^*]_n \right) \]

\[ = \mu [\tilde{S}\tilde{S}^T]_{i,j} \left( \sum_{m=1}^{N} \sum_{n=1}^{N} [\tilde{S}\tilde{S}^T]^*_{m,n} [w_1]_m [w_1^*]_n \right) + \nu \sum_{\alpha=1}^{K} [\tilde{s}_\alpha]_i [\tilde{s}_\alpha^*]_j \sum_{m=1}^{N} [\tilde{s}_\alpha]_m [w_1]_m \sum_{n=1}^{N} [\tilde{s}_\alpha]_n [w_1^*]_n. \]

Writing (122) in matrix form, we have

\[ M \cdot E \{ \Delta C^r w_1^H \Delta C^r \} = \mu \left( w^H_1 (\tilde{S}\tilde{S}^T)^* w_1 \right) \tilde{S}\tilde{S}^T + C^r w_1^H C^r + \nu \tilde{S} \tilde{D} \tilde{S}^T, \quad (123) \]

with \( \tilde{D} \triangleq \text{diag} \{ (\tilde{s}_1^H w_1)^2, (\tilde{s}_2^H w_1)^2, \ldots, (\tilde{s}_K^H w_1)^2 \} \).
Substituting (123) into (118), we get

$$
\tilde{C}_w = \mu \left[w_1^T \left(\tilde{S}\tilde{S}^T\right)^* w_1\right] C_r^{-1} \tilde{S} \tilde{S}^T C_r^{-1} + w_1 w_1^T + \nu C_r^{-1} \tilde{S} \tilde{S}^T C_r^{-1}
$$

**Subspace Blind Detector** - The proof follows a similar line as that of Proposition 1 in [5], and we will therefore abbreviate some steps. Consider the function \((\hat{U}_s, \hat{A}_s) \to \hat{w}_1 = \hat{U}_s \hat{A}_s^{-1} \hat{U}_s^H \hat{s}_1\). By Lemma 3, \(M (\hat{w}_1 - w_1)\) is asymptotically complex Gaussian as \(M \to \infty\), with zero-mean and covariance matrices given by

$$
C_w \triangleq M \cdot E \{\Delta w_1 \Delta w_1^H\}, \quad \text{and} \quad \tilde{C}_w \triangleq M \cdot E \{\Delta w_1 \Delta w_1^T\},
$$

where \(\Delta w_1\) is the differential of \(\hat{w}_1\) at \((U_s, A_s)\). Denote \(U = [U_s \ U_n]\). Define

$$
T \triangleq U^H \tilde{C}_r U = U^H \left(\hat{U}_s \hat{A}_s \hat{U}_s^H + \hat{U}_n \hat{A}_n \hat{U}_n^H\right) U.
$$

(124)

Since \(T\) is a unitary transformation of \(\tilde{C}_r\), its eigenvalues are the same as those of \(\tilde{C}_r\). Hence its eigendecomposition can be written as

$$
T = W_s \hat{A}_s W_s^H + W_n \hat{A}_n W_n^H,
$$

(125)

where \(W = [W_s \ W_n] \triangleq U^H \left[\hat{U}_s \ \hat{U}_n\right] U\) contains eigenvectors of \(T\). Then we have

$$
\Delta w_1 = \Delta \left(\hat{U}_s \hat{A}_s^{-1} \hat{U}_s^H\right) \hat{s}_1 = U \Delta \left(W_s \hat{A}_s^{-1} W_s^H\right) U^H \hat{s}_1.
$$

(126)

Note that in general in order to invoke Lemma 4, the diagonal elements of \(W_s\) should be positive. However in (126), since the quantity of interest is \(\Delta \left(W_s \hat{A}_s^{-1} W_s^H\right), [W_s]_{i,i}\) can always treated as positive. The differential in (126) at \((I_N, A_s)\) is given by

$$
\Delta \left(W_s \hat{A}_s^{-1} W_s^H\right) = \Delta W_s \hat{A}_s^{-1} E_s^H + E_s \hat{A}_s^{-1} \Delta W_s^H - E_s \hat{A}_s^{-2} \Delta \hat{A}_s E_s^H,
$$

(127)

where \(E_s\) is the first \(K\) columns of \(I_N\). Using Lemma 4, after some manipulations, we have [5],

$$
[z]_i = \left[-\frac{1}{\lambda_i} \sum_{k=1}^{K} \frac{1}{\lambda_k} [\Delta T]_{i,k}[U^H \hat{s}_1]_k\right] \delta_{i \leq K} + \left[\sum_{k=1}^{K} \frac{1}{\lambda_k(\lambda_k - \eta)} [\Delta T]_{i,k}[U^H \hat{s}_1]_k\right] \delta_{i > K}.
$$

(128)

Since \(E \{\Delta T\} = 0\), by Lemma 2, for \(1 \leq i, j \leq N\),

$$
M \cdot E \{[\Delta T]_{i,k}, [\Delta T]_{j,l}\} = \lambda_i \lambda_k \delta_{i = j} \delta_{k = j} + \mu \left[U^H \hat{s} \hat{s}^T U^*\right]_{i,j} \left[U^H \hat{s} \hat{s}^T U^*\right]_{k,l}
$$

$$
+ \nu \sum_{\alpha=1}^{K} \left[U^H \hat{s}_\alpha\right]_i \left[U^H \hat{s}_\alpha\right]_j \left[U^H \hat{s}_\alpha\right]_k \left[U^H \hat{s}_\alpha\right]_l.
$$

(129)
Using (128) and (129), we have for $i, j \leq K$

$$M \cdot E \left\{ [zz^H]_{i,j} \right\} = \frac{1}{\lambda_i \lambda_j} \sum_{k=1}^{K} \sum_{l=1}^{K} \left( \frac{1}{\lambda_k \lambda_l} E \left\{ [\Delta T]_{i,k}[\Delta T]^*_j \right\} [U^H \tilde{s}_1]_k [U^H \tilde{s}_1]^*_l \right) = T_a + T_b + T_c, \quad (130)$$

where $T_a$, $T_b$, and $T_c$ correspond to the three terms in (129). We can then calculate

$$T_a = \frac{\delta_{i,j}}{\lambda_j} \sum_{k=1}^{K} \sum_{l=1}^{K} \sum_{k=1}^{K} \left( \frac{1}{\lambda_k \lambda_l} [U^H \tilde{s}_1]_k [U^H \tilde{s}_1]^*_l \right) = \frac{\delta_{i,j}}{\lambda_i} \sum_{k=1}^{K} \frac{1}{\lambda_k} \left( [U^H \tilde{s}_1]_k [U^H \tilde{s}_1]^*_k \right), \quad (131)$$

$$T_b = \frac{\mu}{\lambda_i \lambda_j} \sum_{k=1}^{K} \sum_{l=1}^{K} \sum_{k=1}^{K} \left( \frac{1}{\lambda_k \lambda_l} [U^H \tilde{s}_1]_k [U^H \tilde{s}_1]^*_k \right) \frac{1}{\lambda_k \lambda_l} \left( [U^H \tilde{s}_1]_k [U^H \tilde{s}_1]^*_l \right), \quad (132)$$

$$T_c = \frac{\nu}{\lambda_i \lambda_j} \sum_{k=1}^{K} \sum_{l=1}^{K} \sum_{k=1}^{K} \sum_{l=1}^{K} \left( \frac{1}{\lambda_k \lambda_l} [U^H \tilde{s}_1]_k [U^H \tilde{s}_1]^*_k \right) \frac{1}{\lambda_k \lambda_l} \left( [U^H \tilde{s}_1]_k [U^H \tilde{s}_1]^*_l \right), \quad (133)$$

Notice that the last two terms in (129) disappear if at least one of $i, j, k, l$ is larger than $K$. From this it can be seen that $M \cdot E \left\{ [zz^H]_{i,j} \right\} = 0$ for $i > K, j \leq K$ or $i \leq K, j > K$, and that for $i, j > K$ we get

$$M \cdot E \left\{ [zz^H]_{i,j} \right\} = \sum_{k=1}^{K} \sum_{l=1}^{K} \frac{\lambda_i \lambda_k \delta_{i,j} \delta_{k,l}}{\lambda_k (\lambda_k - \eta) \lambda_l (\lambda_l - \eta)} [U^H \tilde{s}_1]_k [U^H \tilde{s}_1]^*_k \eta \delta_{i,j} \sum_{k=1}^{K} \frac{1}{\lambda_k (\lambda_k - \eta)^2} [U^H \tilde{s}_1]_k [U^H \tilde{s}_1]^*_k, \quad (134)$$

Writing (131) through (134) in matrix form and using that $M \cdot E \left\{ \Delta w_1 \Delta w_1^H \right\} = M \cdot UE \left\{ zz^H \right\} U^H$, we obtain (33). The symmetric covariance matrix (34) is obtained through very similar steps. □

**Appendix C: Proof of Theorem 2**

The blind detector with blind channel estimation is expressed as $\hat{w}_1 = \hat{U}_s \hat{A}^{-1}_s U^H_s S_1 \hat{h}_1$. Hence its differential is given by

$$\Delta w = \Delta \left( U_s \hat{A}^{-1}_s U^H_s \right) \tilde{s}_1 + U_s \hat{A}^{-1}_s U^H_s S_1 \Delta \hat{h}_1 \quad (135)$$

The first differential in (135) has been calculated in Appendix B and is given by (126) and (128). We next calculate the second differential in (135). As in the proof of Theorem 1 (subspace case),
we will apply the coordinate transformation $U$ and express the results in terms of the eigenvectors of $T$ given by (124). First, since

$$\hat{Q} \triangleq S_1^H \hat{U}_n \hat{U}_n^H S_1 = S_1^H \left( I_N - \hat{U}_s \hat{U}_s^H \right) S_1,$$

its differential is then given by

$$\Delta Q = -S_1 \Delta \left( \hat{U}_s \hat{U}_s^H \right) S_1 = - \left( U^H S_1 \right) \Delta \left( \hat{W}_s \hat{W}_s^H \right) \left( U^H S_1 \right)$$

where $\Delta W_s$ can be found from Lemma 4 as

$$\Delta W_{i,k} = \begin{cases} 0, & i = k \\ \lambda_k - \lambda_i, & i \neq k \end{cases}, \quad i = 1, \ldots, N, \quad k = 1, \ldots, K.$$  

Next, since $\hat{h}_1$ is the minimum eigenvector of $\hat{Q}$, we use Lemma 4 again to express $\Delta h_1$ in terms of $\Delta Q$. Let the eigendecomposition of the $L \times L$ matrix $Q$ be

$$Q = U_q \Lambda_q U_q^H,$$

where by assumption the minimum eigenvalue $\lambda_{q,L} = 0$. Define

$$\Delta X \triangleq U_q^H \Delta Q U_q,$$

where $v$ denote the $L$-th eigenvector of $U_q^H Q U_q$. Then we have by Lemma 4

$$\Delta h_1 = U_q \Delta v,$$

$$[\Delta v]_i = \begin{cases} 0, & i = L \\ -\frac{1}{\lambda_i - \lambda_{q,L}} [\Delta X]_{i,L}, & i \neq L \end{cases}.$$

Now substitute (137) and (140) into (141) and (142), we obtain

$$[\Delta X]_{i,L} = U_q^H \Delta Q u_{q,L} = -U_q^H \left( U^H S_1 \right)^H \Delta W_s E^T_s \left( U^H \hat{s}_1 \right) - U_q^H S_1^H U_s E_s \Delta W_s^H \left( U^H \hat{s}_1 \right),$$

$$\Delta h_1 = U_q \Lambda q_u \Lambda q_u^{-1} U_q^H \left( U^H S_1 \right)^H \left[ \Delta W_s E^T_s \left( U^H \hat{s}_1 \right) + E_s \Delta W_s^H \left( U^H \hat{s}_1 \right) \right] \frac{Q^q}{y},$$

where we have used the fact that $\hat{s}_1 = S_1 u_{q,L}$ in (143); and in (144) $\Lambda q_u$ denotes the submatrix of $\Lambda_q$ which contains the $(L - 1)$ non-zero eigenvalues of $Q$, and $U_q$ denotes the corresponding

$^2$Note that differential given by Lemma 4 is for $[v]_L > 0$. This specific phase normalization carried through the rest of the expressions results in the phase factor in Theorem 2.
This gives Corollary 1 and the second term in (53).

appendix D: Some Derivations

Derivation of (59) and (60): Using (2) and (53), we have

\[
\text{tr}(C_w C_r) = (C_w^0 C_r) + \xi_1 \text{tr}(I),
\]

(153)
where using (55)
\[
\text{tr}(\Gamma) \triangleq \text{tr} \left[ \left( \Psi Q^i \Psi^H \right) (U_s A_s U_s^H) \right] = \text{tr} \left[ (S_s U_s A_s^{-1} U_s^H S_1) Q^i \right].
\]  
(154)

Finally, \(\beta_1\) in (56) can be computed as follows. Denote \(\tilde{R} \triangleq \tilde{S}^H \tilde{S}\), then
\[
\beta_1 = \eta \tilde{s}_1^H U_s A_s U_s^H U_s (A_s - \eta I_K)^{-1} U_s^H U_s (A_s - \eta I_K)^{-1} U_s^H \tilde{s}_1
\]
\[
= \eta e_1^T \tilde{S}^H (U_s A_s U_s^H) \tilde{S} \tilde{R}^{-2} e_1 = \eta e_1^T \tilde{S}^H (U_s A_s U_s^H + \eta U_n U_n^H) \tilde{S} \tilde{R}^{-2} e_1
\]
\[
= \eta e_1^T \tilde{S}^H (\tilde{S} \tilde{S}^H + \eta I_N) \tilde{S} \tilde{R}^{-2} e_1 = \eta \left[ I_K + \eta \tilde{R}^{-1} \right]_{1,1}.
\]  
(155)

References


