(a) yes: \( f_X(x) = xe^{-x}, f_Y(y) = e^{-y}, 0 < x < \infty, 0 < y < \infty \)

(b) no: 
\[
\int_{x}^{1} f(x, y) \, dy = 2(1-x), 0 < x < 1
\]
\[
f_Y(y) = \int_{0}^{y} f(x, y) \, dx = 2y, 0 < y < 1
\]

(a) No, since the joint density does not factor.

(b) \( f_X(x) = \int_{0}^{1} (x + y) \, dy = x + 1/2, \ 0 < x < 1. \)

(c) \( P\{X + Y < 1\} = \int_{0}^{1} \int_{0}^{1-x} (x + y) \, dy \, dx \)
\[
= \int_{0}^{1} [x(1-x) + (1-x)^2 / 2] \, dx = 1/3
\]
For $j = i$: $P(Y = i \mid X = i) = \frac{P(Y = i, X = i)}{P(X = i)} = \frac{1}{36P(X = i)}$

For $j < i$: $P(Y = j \mid X = i) = \frac{2}{36P(X = i)}$

Hence

$$1 = \sum_{j=1}^{i} P(Y = j \mid X = i) = \frac{2(i-1)}{36P(X = i)} + \frac{1}{36P(X = i)}$$

and so, $P(X = i) = \frac{2i-1}{36}$ and

$$P(Y = j \mid X = i) = \begin{cases} 1 & j = i \\ \frac{2i-j}{2} & j < i \end{cases}$$

(a) First obtain marginal for $Y$: $p_Y(1) = \frac{1}{4}$, $p_Y(2) = \frac{3}{4}$.

Then we have that $p_{X\mid Y}(1\mid 1) = p_{X\mid Y}(2\mid 1) = \frac{1}{2}$.

And $p_{X\mid Y}(1\mid 2) = \frac{1}{3}$ and $p_{X\mid Y}(2\mid 2) = \frac{2}{3}$.

(b) Obtain marginal for $X$: $p_X(1) = \frac{3}{8}$ and $p_X(2) = \frac{5}{8}$.

Note that $p_{X,Y}(i,j) \neq p_X(i)p_Y(j)$ so $X$ and $Y$ are not independent.

c) $P(XY \leq 3) = \frac{1}{2}$, $P(X+Y > 2) = \frac{7}{8}$.

$P(X=1\mid Y>1) = \frac{1}{3}$ and $P(X=2\mid Y>1) = \frac{2}{3}$. 
2) This problem is similar to the practice exam problem.

a) We have \( P(X = k) = \prod_{i=1}^{k-1} \frac{(3-i+1)(6)}{(9-i+1)(9-k+1)} \) for \( k = 1, \ldots, 4 \). Therefore \( P(X = 1) = \frac{2}{3}, P(X = 2) = \frac{1}{4}, P(X = 3) = \frac{1}{14}, P(Y = 4) = \frac{1}{84} \). To compute the joint pmf of \( X \) and \( Y \) we need to compute conditional pmf given by

\[
P(Y = j|X = k) = \prod_{i=1}^{j-1} \frac{(4-i-k)(5)}{(8-i+1)(8-j+1)}, \quad 1 \leq j \leq 5-k, k = 1, 2, 3, 4
\]

The joint pmf is given by

\[
p_{X,Y}(j, k) = P(Y = j|X = k)P(X = k)
\]

where

\[
p_{X,Y}(1, 1) = \frac{5}{12}, p_{X,Y}(1, 2) = p_{X,Y}(2, 1) = \frac{5}{28}
\]

\[
p_{X,Y}(3, 1) = p_{X,Y}(2, 2) = p_{X,Y}(1, 3) = \frac{5}{84}
\]

\[
p_{X,Y}(4, 1) = p_{X,Y}(3, 2) = p_{X,Y}(2, 3) = p_{X,Y}(1, 4) = \frac{1}{84}
\]

The marginal pmf of \( Y \) is the same as \( X \).

b) \( X \) and \( Y \) are not independent.

c) We get that \( m_X = m_Y = 10/7, \text{VAR}(X) = \text{VAR}(Y) = 45/98, \text{E}(XY) = 55/28 \), and \( \text{COV}(X, Y) = -15/196 \).

d) To simulate on matlab, first generate a permutation of the numbers 1 through 9. Then we can simulate drawing balls from an urn without replacement. Matlab code is below.

```matlab
n=10000; k=1:9; times=zeros(6,n);
u9= rand(9,n);
[a,perm] = sort(u9);
sample = [ones(1,6) zeros(1,3)];
blue = sample(perm);
for i=1:n,
times(:,i) = k(blue(:,i)==1)';
end
sojourn=diff([zeros(1,n);times]);
x=sojourn(1,:); y=sojourn(2,:);
mx=mean(x);my=mean(y); sx=var(x);sy=var(y);
cov=mean(x.*y) -mx*my; rho=cov/(sqrt(sx*sy));
```

The averages computed on matlab are 1.4277 for mean of \( X \), 1.4298 for mean of \( Y \), .4586 for variance of \( X \), .4599 for variance of \( Y \), and -.0718 for covariance of \( X \) and \( Y \). The correlation coefficient is -.1564.
3)

a) This was done in class. There are two methods. You can either either use moment generating functions or convolve pdfs. We have that $M_Z(t) = \lambda^2/(\lambda - t)^2$. This is MGF of a Gamma RV with parameter $k = 2$ with pdf $p_Z(z) = \lambda^2 z \exp(-\lambda z)u(z)$.

b) Again can use MGF or work with pdfs. Note that

$$M_W(t) = \mathbb{E}(\exp((X-Y)u)) = M_X(t)M_Y(-t) = \lambda^2/(\lambda^2-t^2) = .5\lambda/(\lambda-t) + .5\lambda/(\lambda+t)$$

We can take inverse transform to get that $p_W(w) = .5\lambda \exp(-\lambda|w|)$.

c) Here we work with CDF. We have that

$$F_V(v) = P(V \leq v) = P(\max(X,Y) \leq v) = P(X \leq v, Y \leq v) = (1 - \exp(-\lambda v))u(v).$$

To find pdf differentiate CDF to get that $p_V(v) = 2\lambda \exp(-\lambda v)(1 - \exp(-\lambda v))u(v)$.

To simulate using matlab generate two exponential RVs and perform function. lambda = 1; n=100000;
x = -1/lambda *log(rand(1,n)); y = -1/lambda *log (rand(1,n));
z=x+y, w=x-y; v = max(x,y);

![Graphs](image.png)

Figure 1: pdf of Z
Figure 2: pdf of $W$

Figure 3: pdf of $V$