Equivalence of BIBO stable and impulse response being absolutely integrable

A system is Bounded Input Bounded Output (BIBO) stable if bounded inputs yield bounded outputs. Consider a linear time invariant system with impulse response \( h(t) \). In this case we have the following result.

**Statement:** The LTI system with impulse response \( h(t) \) is BIBO stable if and only if \( h(t) \) is absolutely integrable (i.e. \( \int_{-\infty}^{\infty} |h(\tau)|d\tau \) is finite).

**Proof:** If \( h(t) \) is absolutely integrable, then assume that input is bounded, \( |x(t)| < B_X \). Then we have that

\[
|y(t)| = |\int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau| \\
\leq \int_{-\infty}^{\infty} |h(\tau)x(t-\tau)|d\tau \leq \int_{-\infty}^{\infty} |h(\tau)||x(t-\tau)|d\tau \\
\leq B_X \int_{-\infty}^{\infty} |h(\tau)|d\tau \leq B_Y
\]

with the last equality due to the fact that \( h(t) \) is absolutely integrable, therefore \( y(t) \) is bounded and LTI system is BIBO stable.

If \( h(t) \) is not absolutely integrable let \( x(t) = \text{sgn}(h(-t)) \) where the \text{sgn} function is 1 when the argument is positive, -1 when the argument is negative, and zero when the argument is 0. Here we will show that the output \( y(t) \) is unbounded when we choose a bounded input given by \( x(t) \). Note that

\[
y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau
\]

and that the output at \( t = 0 \) is given by

\[
y(0) = \int_{-\infty}^{\infty} h(\tau)\text{sgn}(h(-\tau))d\tau = \int_{-\infty}^{\infty} |h(\tau)|d\tau = \infty
\]

Here we have a bounded input that yields an unbounded output and therefore LTI system is not BIBO stable.