More on counting

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There is one more counting technique you need to be familiar with. Again, note that we always can use one of the template problems we discussed in class.

HW 1 We will count the number of ways of choosing a vector of numbers \((n_1, n_2, \ldots, n_m)\), such that for all \(i\), \(n_i \geq 1\) and \(\sum_i n_i = n\).

1. Consider one possible \((n_1, \ldots, n_m)\). Let \(S_1 = n_1\), \(S_2 = n_1 + n_2\), and in general, \(S_r = n_1 + \ldots + n_r\).

Can \(S_i\) and \(S_j\) be equal if \(i \neq j\)?

2. What is \(S_m\)?

3. Given \(S = (S_1, \ldots, S_m)\), can you obtain \(n = (n_1, \ldots, n_m)\)? How?

4. Since there is exactly one way to get \((S_1, \ldots, S_m)\) from \((n_1, \ldots, n_m)\) and vice versa, counting one is same as the other. We say there is a bijection between \(S\) and \(n\).

Show that \# ways of choosing \((S_1, \ldots, S_m)\) is simply \(\binom{n-1}{m-1}\). (Hint: each choice of \(S_1, \ldots, S_m\) is simply a set of \(m\) numbers, but the last number is always \(n\)).

Therefore, the \# ways of choosing a vector of numbers \((n_1, n_2, \ldots, n_m)\), such that for all \(i\), \(n_i \geq 1\) and \(\sum_i n_i = n\) is \(\binom{n-1}{m-1}\).

To count \# ways of picking \(t = (t_1, \ldots, t_m)\) such that \(t_i \geq 0\) and \(\sum_i t_i = n\), consider the vector \(t + 1 = (t_1 + 1, \ldots, t_m + 1)\). Again there is exactly one way to obtain \(t\) from \(t + 1\) and vice versa. But we know how to count \(t + 1\).

1. What do the components of \(t + 1\) add up to?

2. Show that \# ways of picking \(t + 1\) and therefore \(t\) is \(\binom{n+m-1}{m-1}\).

HW 2 \# ways of picking among \(n\) objects, \(k\) trials, with replacement, no ordering implied. We had postponed this case for later. An outcome is a multiset \(\{1, 1, \ldots, n-1\}\) for example. (A multiset is simply a collection of numbers, possibly repeated, with no ordering implied among them).

We see each outcome as a vector of \(n\) numbers, \((t_1, \ldots, t_n)\), the \(i\)th number \(t_i\) denoting how many times \(i\) shows up in the \(k\) trials done. For example, consider \(k = 3\) and \(n = 4\), and consider the outcome \(\{1, 1, 4\}\).

We represent the outcome \(\{1, 1, 4\}\) by \((2, 0, 0, 1)\), namely that 1 appears 2 times, 4 appears once, and 2 and 3 do not appear at all.

In communication theory, this is usually called the type of the outcome.

1. What is \(t_1 + t_2 + \ldots + t_n\)?

2. Is the type \((1, 0, 0, 2)\) different from \((1, 2, 0, 0)\) or \((2, 0, 0, 1)\)? Namely, do they represent different outcomes? Do different outcomes represent different types? Therefore, is counting the number of outcomes same as counting the number of types of outcomes?

3. Using the previous problem, what is the number of outcomes?