The Wideband Slope of Interference Channels: The Infinite Bandwidth Case

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Abstract

It is well known that minimum received energy per bit $\frac{E_b}{N_0}$ in the interference channel is $-1.59dB$ as if there were no interference. Thus, the best way to mitigate interference is to operate the interference channel in the low power regime, that is in the limit of infinite bandwidth. However, when the bandwidth is large, but finite, $\frac{E_b}{N_0}$ alone does not characterize performance. Verdu introduced the wideband slope $S_0$ to characterize the performance in this regime. We show that a wideband slope of $\frac{S_0}{S_{0, \text{no interference}}}$ is achievable. This result is similar to recent results on degrees of freedom in the high SNR regime, and we use a type of interference alignment using delays to obtain the result. We also show that in many cases the wideband slope is upper bounded by $\frac{S_0}{S_{0, \text{no interference}}}$ for large number of users $K$.

Index Terms

Interference channels, wideband slope, interference alignment.

I. INTRODUCTION

RECENTLY there has been much interest in interference channels[1], [2], [3]. In [4] it was shown that in the high-SNR regime, it is possible to achieve $K/2$ degrees of freedom in a $K$-user interference channel (half of the $K$ degrees of freedom if there were no interference). The basic idea is to align interference from all $K-1$ undesired users in half the signal space, and then receive the desired signal in the other half space without interference, an idea pioneered by [5]. The paper [4] has inspired a large body of research on interference alignment in the high-SNR regime, for example [6], [7], [8], [9], [10], [11].

In this paper we consider the interference channel in the low power regime (low-SNR regime). While the work in [4] and follow-up work shows impressively that much can be done to mitigate the effect of interference in the high-SNR regime, one could argue that the best way to mitigate the effect of interference is to avoid the high-SNR regime and instead operate in the low-SNR regime, when possible. It is well-known (e.g., [12]) that in a point-to-point link the received minimum energy per bit $\frac{E_b}{N_0}$ is $-1.59dB$ is achieved as the spectral efficiency\(^1\) (bits/s/Hz) $R \rightarrow 0$. It also known [12] that this energy is unchanged in the presence of interference. Thus, in this limit the effect of interference is completely eliminated. However, as Verdu pointed out in [13], in practical systems the spectral efficiency must be non-zero, though it might be still small. One way to characterize the effect of this is through the wideband slope. The wideband slope is defined by

$$
S_0 \triangleq \lim_{N \rightarrow \infty} \frac{E_b}{N_0} \frac{C \left( \frac{E_b}{N_0} \right)}{10 \log_{10} E_b N_0 - 10 \log_{10} E_b N_0^\text{min} - 10 \log_{10} 2} \tag{1}
$$

where $C \left( \frac{E_b}{N_0} \right)$ is the spectral efficiency as a function of $\frac{E_b}{N_0}$. The wideband slope essentially represents a second order approximation of the capacity in the low power regime, or first order approximation of the spectral efficiency. For example, we can write

$$
C \approx \frac{S_0}{10 \log_{10} 2} \left( 10 \log_{10} \frac{E_b}{N_0} - 10 \log_{10} \frac{E_b}{N_0} \right) \approx 10 \log_{10} \frac{E_b}{N_0} + \frac{C}{S_0} 10 \log_{10} 2
$$

Examples in [13] show that this is a good approximation for many channels up to fairly high spectral efficiencies, e.g., 1 bit/s/Hz.

The reference point for wideband slope is the point-to-point AWGN (additive white Gaussian noise) channel, which has a wideband slope of 2. The wideband slope also characterizes the bandwidth required to transmit at a given rate (in the low

\(^1\)Consistent with [13] we use spectral efficiency for the capacity in bits/s/Hz, and rate for the capacity in bits/s.

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power regime). For example, if the wideband slope is decreased from 2 to 1, twice the bandwidth is required for transmitting at a given rate.

The wideband slope for interference channels was considered for the two-user channel in [14] (a generalization to QPSK can be found in [15]). They showed that TDMA is not efficient in the low power regime. In Section III we will extend the results of [14]. However, the main focus of the paper is the $K$-user channel, and in particular how interference alignment as in [4] can be used in the low power regime.

Traditional interference alignment as in [4] does not work in the low power regime. The results in [4] depend on time or frequency selectivity of the channel. However, to achieve the minimum energy per bit in a non-flat channel, all data needs to be transmitted on the strongest channel only – which means that the wideband slope is poor (e.g., $\frac{2}{K}$ for a $K$-user interference channel). On the other hand, delay differences between different paths can be effectively used. Delay differences for interference alignment was also considered in [16], [17]. However, delay is a more natural fit for the low power regime. Namely, as the bandwidth $B \to \infty$ even the smallest delay will eventually be magnified to the point of being much larger than the symbol duration. Therefore, delays can be efficiently manipulated and used for high bandwidth.

In this paper we will prove that interference alignment using delays can be used to achieve half the wideband slope of an interference-free channel, similar to losing half the degrees of freedom in the high-SNR regime. We will also show that generally it is difficult to obtain a larger wideband slope. The fact that wideband slope is reduced by only half means that near single-user performance can be obtained in the low-power regime. For example, if it is desired to transmit at $R = 0.5$ spectral efficiency, in the interference-free channel this requires $0.6dB$ extra energy over the minimum energy per bit for $R = 0$. With interference, $1.2dB$, e.g., $0.6dB$ extra energy is needed to overcome interference, independently of the number of users.

II. System Model

We consider a $K$-user interference channel. There are $K$ transmitters, numbered 1 to $K$, and $K$ receivers, also numbered 1 to $K$. Transmitter $i$ needs to transmit a message to receiver $i$, but has no need for any other messages. All nodes have one antenna. As the specifics of the wireless model affect the results, we will discuss in more details the physical modelling of the system. The nodes are placed in a two or three dimensional space, where the distance from transmitter $i$ to receiver $j$ is denoted $d_{ij}$. We assume the wireless signal propagates directly from node $i$ to node $j$, and the delay in signal arrival is therefore determined by $d_{ij}$, a line-of-sight model.

Consider at first a single transmitter receiver pair, $i$ and $j$. Let the complex discrete-time transmitted signal of transmitter $i$ be $x_i[n]$ and the corresponding base band (continuous-time) signal be $x_i(t)$ with (complex) bandwidth $B$. This is modulated with the carrier signal $c(t) = \exp(\iota \omega_0 (t - \varsigma_i))$, where $\omega_0$ is the carrier frequency and $\varsigma_i$ is the delay (phase offset) in the oscillator at transmitter $i$ (and $\iota = \sqrt{-1}$). The real part is transmitted,

\[
s_i(t) = \Re \{ \exp(\iota \omega_0 (t - \varsigma_i)) x_i(t) \} = \cos(\omega_0 (t - \varsigma_i)) \Re \{ x_i(t) \} - \sin(\omega_0 (t - \varsigma_i)) \Im \{ x_i(t) \}
\]

The received signal at receiver $j$ is

\[
y_j(t) = A_{ji} \Re \{ \exp(\iota \omega_0 (t - \varsigma_i - \tau_{ji})) x_i(t - \tau_{ji}) \} + \bar{z}_j(t)
\]

where $A_{ij}$ is an attenuation factor, $c$ is the speed of light, and $\bar{z}_j(t)$ is white Gaussian noise with power spectral density $N_0$. This is modulated to base band by multiplying with $\exp(\iota \omega_0 (t - v_j))$, where $v_j$ is the delay in the oscillator at node $j$, and using a lowpass filter, resulting in the base band signal

\[
y_j(t) = A_{ji} \exp(\iota \omega_0 (\varsigma_i + \tau_{ji} - v_j)) x_i(t - \tau_{ji}) + z_j(t)
\]

This expression is valid on the assumption that $\omega_0 > B$. Here $z_j(t)$ is complex circular white Gaussian noise of bandwidth $B$.

Return now to the interference channel. When all nodes transmit, the received signal at receiver $j$ is

\[
y_j(t) = A_{jj} \exp(\iota \omega_0 (\varsigma_j + \tau_{jj} - v_j)) x_j(t - \tau_{jj}) + \sum_{i \neq j} A_{ji} \exp(\iota \omega_0 (\varsigma_i + \tau_{ji} - v_j)) x_i(t - \tau_{ji}) + z_j(t)
\]

This is sampled at the Nyquist frequency $f_s = \frac{1}{2B}$ (as $B$ is the complex bandwidth). Let

\[
n_{ij} = \lfloor \tau_{ij} B + \frac{1}{2} \rfloor \quad (2)
\]

\[
\delta_{ij} = \tau_{ij} B - \lfloor \tau_{ij} B + \frac{1}{2} \rfloor \quad (3)
\]

where $\lfloor x \rfloor$ is the largest integer smaller than or equal to $x$. Without loss of generality we can assume that the received signal at receiver $j$ is sampled symbol-synchronous with the desired signal. Then the discrete-time model is

\[
y_j[n] = A_{jj} \exp(\iota \omega_0 (\varsigma_j + \tau_{jj} - v_j)) x_j[n - n_{jj}] + \sum_{i \neq j} A_{ji} \exp(\iota \omega_0 (\varsigma_i + \tau_{ji} - v_j)) \tilde{x}_i[n - n_{ji}] + z_j[n]
\]

\[
y_j[n] = A_{jj} \exp(\iota \omega_0 (\varsigma_j + \tau_{jj} - v_j)) x_j[n - n_{jj}] + \sum_{i \neq j} A_{ji} \exp(\iota \omega_0 (\varsigma_i + \tau_{ji} - v_j)) \tilde{x}_i[n - n_{ji}] + z_j[n]
\]
where
\[ \tilde{x}_i[n] = \sum_{m=-\infty}^{\infty} x_i[m] \text{sinc}(n - m + \delta_{ij}) \] (4)

We will also occasionally make the dependency on the fractional delay explicit as follows
\[ \tilde{x}_i[n, \delta_{ji}] = \sum_{m=-\infty}^{\infty} x_i[m] \text{sinc}(n - m + \delta_{ij}) \] (5)

By the Shannon sampling theorem, this discrete-time model is equivalent with the original continuous-time model. Results do not change if we normalize the time at each receiver so that \( n_{ij} = 0 \). And as the carrier frequency is large, the phases \( \exp(i\omega_0(\xi_i + \tau_{ji} - v_{ij})) \) can be reasonably modelled as independent uniform random variables \( \theta_{ji} \) over the unit circle. We therefore arrive at the following expression for the received signal
\[ y_j[n] = A_{jj} \exp(i\theta_{jj}x_j[n] + \sum_{i \neq j} A_{ji} \exp(i\theta_{ji}\tilde{x}_i[n - n_{ji}] + z_j[n]) \]
\[ = C_{jj}x_j[n] + \sum_{i \neq j} C_{ji}\tilde{x}_i[n - n_{ji}] + z_j[n] \] (6)

where \( C_{ji} = A_{ji} \exp(i\theta_{ji}) \).

Notice that this model makes no assumptions on or approximations of modulation, i.e., assuming rectangular waveforms. Transmission is therefore strictly bandlimited to a bandwidth \( B \), as opposed to \( [16] \).

What is interesting is that there are two distinct realizations of this model depending on how the low power regime is approached. Compare with a point-to-point AWGN channel with a rate
\[ R = B \log\left(1 + \frac{P}{BN_0}\right) \]
The low power results are based on a Taylor series of \( \log(1 + x) \): therefore as long as \( \frac{P}{BN_0} \to 0 \) low-power results such as minimum energy per bit and wideband slope are valid. The key is that the spectral efficiency \( R = R/B \to 0 \), not that \( B \to \infty \). For the interference channel, on the other hand, different results are obtained depending on whether \( B \to \infty \) or \( \frac{P}{BN_0} \to 0 \) while \( B < \infty \). We call the former case the infinite bandwidth case, and this is the topic of this paper; the finite bandwidth case will be considered in a later paper.

A. Performance criteria

In [14] the whole slope region of the interference region in the two user case was analyzed. However, for more than two users it is complicated to compare complete slope regions, and we are therefore looking at a single quantity to characterize performance. We consider two different scenarios

- **The equal power case.** In this case we maximize the sum rate \( R_s = R_1 + R_2 + \cdots + R_K \) under the constraint \( P_1 = P_2 = \cdots = P_K \). We want to characterize the wideband slope of the sum rate.

- **The equal rate case.** In this case we minimize the total power \( P = P_1 + P_2 + \cdots + P_K \) under the constraint \( R = R_1 = R_2 = \cdots = R_K \). We want to characterize the wideband slope of the sum rate \( R_s = KR \).

The first case could correspond to a scenario where each node needs to consume energy at the same rate, e.g., so that batteries last the same for all nodes. The second case could correspond to a scenario where we want to minimize total system energy consumption. Each case can be easily generalized to unbalanced cases, e.g. \( \mu_1P_1 = \mu_2P_2 = \cdots = \mu_KP_K \), but we only consider the balanced case here to keep results concise.

As performance measure we use
\[ \Delta S_0 = \frac{S_0}{S_{0, \text{no interference}}} \]

The quantity \( S_{0, \text{no interference}} \) is the wideband slope of a \( K \)-user interference channel where all interference links are nulled, \( |C_{ij}| = 0, i \neq j \), but the direct links \( C_{ii} \) are unchanged. We can interpret \( \Delta S_0 \) as the loss in wideband slope due to interference, or equivalently \( (\Delta S_0)^{-1} \) as (approximately) the additional bandwidth required to overcome interference. Alternatively, if we define
\[ \Delta E_b = 10 \log_{10} \frac{E_b}{N_0} - 10 \log_{10} \frac{E_b}{N_{0, \text{min}}} \]
as the extra energy required to operate at a spectral efficiency \( R > 0 \), we have
\[ \Delta E_b \approx (\Delta S_0)^{-1} \Delta E_{b, \text{no interference}} \]
for small increases in spectral efficiency. Thus, \( (\Delta S_0)^{-1} \) also measures the amount of energy needed to overcome interference.
III. The Two User Case

We will start by analyzing the two user case as this is instrumental for the \(K\)-user case. It turns out that all results in the two user case are independent of delay, thus independent of how the low power regime is approached. This indicates that the capacity region in the two user case could be independent of delay in general, but we have not been able to prove so.

First we will outline the strategies that can be used to obtain an achievable rate

- **Interference decoding.** If \( |C_{j1}| > |C_{i1}| \) user \( j \) can decode the message from user \( i \), and the interference from user \( i \) does not affect the wideband slope for user \( j \) (it’s easy to see that the MAC sum-rate bound at user \( j \) does not affect the wideband slope).
- **Treating interference as noise.** The transmitters use i.i.d Gaussian code books, and the receivers treat the interference as part of the background noise. Notice that delay does not affect the distribution of interference as \( x_i[n - n_{ji}] \) has same distribution as \( x_i[n] \).
- **TDMA.** In time-division multiple access the transmitters use orthogonal time slots. With delay, some care has to be taken in allocating time slots, but this does not affect performance in the limit of large code length.

The best known achievable rate for the interference channel is the Han-Kobayashi region [18]. For the Gaussian interference channel, in particular the idea of transmitting common messages has been shown to be powerful [19]. However, the common message does have a higher \( \frac{E_b}{N_0} \) in the low power limit than the minimum, and therefore does not improve the wideband slope. Let us emphasize in general that

For a bound on rate to be useful as a bound on wideband slope, it needs to have the correct \( \frac{E_b}{N_0} \) in the low power limit.

We will now compare the achievable rates with some outer bounds. The first is a generalization (and restatement) of Theorem 2 of [20] to the channel with delay

**Theorem 1** (Kramer’s bound). Suppose that \( |C_{21}| < |C_{11}| \). Then

\[
R_2 \leq \log \left( 1 + \frac{|C_{22}|^2 P_2 + |C_{21}|^2 P_1}{|C_{11}|^2 2R_1 + 1} \right)
\]

independent of delay.

**Proof:** Put \( C_{12} = 0 \) to enlarge the capacity region. Now assume that, different from the system model (6), receiver 2 also samples the received signal synchronously with the transmitted signal of user 1. We can then write the received signal as

\[
y_1[n] = C_{11} x_1[n] + z_1[n] \\
y_2[n] = C_{22} \hat{x}_2[n - n_{22}] + C_{21} x_1[n] + \tilde{z}_2[n]
\]

where \( \hat{x}_2[n] \) is defined by (4). Now, using degradedness we can further write

\[
y_1[n] = C_{11} x_1[n] + z_1[n] \\
y_2[n] = C_{22} \hat{x}_2[n - n_{22}] + \frac{C_{21}}{C_{11}} y_1[n] + \tilde{z}_2[n]
\]

where \( \tilde{z}_2[n] \) is i.i.d Gaussian noise independent of \( z_1[n] \) with power \( \left( 1 - \frac{|C_{21}|^2}{|C_{11}|^2} \right) N_0 B \). Using Fano’s inequality as usual, we can now bound

\[
nR_2 - n\epsilon_n \leq h(y_2^n) - h(y_2^n | W_2) \\
= h(y_2^n) - h \left( \frac{C_{12}}{C_{11}} y_1^n + \tilde{z}_2^n | W_2 \right)
\]

The second term in (8) is independent of delay, and can be lower bounded by the entropy power inequality [21]. The first term can be upper bounded by the delay-free case. The equation (7) is now obtained just as in the delay-free case. \( \blacksquare \)

**Corollary 2.** Suppose that \( |C_{21}| < |C_{11}| \). In the equal-power case, the wideband slope for the sum rate is upper bounded by

\[
S_0 \leq 2 \frac{(|C_{11}|^2 + |C_{22}|^2)^2}{2|C_{21}|^2 |C_{22}|^2 + |C_{11}|^4 + |C_{22}|^4} \\
\Delta S_0 \leq \frac{1}{2 \left( \frac{|C_{21}|^2 |C_{22}|^2}{|C_{11}|^2 + |C_{22}|^2} + 1 \right)}
\]

independent of delay. Furthermore, if additionally \( |C_{12}| > |C_{22}| \) this bound is achieved by treating interference as noise.

**Proof:** This result can be easily shown given (47) in [20] and (35), (140) in [22].

There is of course a similar result for the symmetric case.
For the equal-rate case if only one interference link is weak, bound (7) does not have the correct $\frac{P_s}{N_0}$ and therefore cannot be used for bounding the wideband slope. If both interference links are weak, we have following corollary.

**Corollary 3.** Suppose that $|C_{21}| < |C_{11}|$ and $|C_{12}| < |C_{22}|$. In the equal-rate case, the wideband slope for the sum rate is upper bounded by

$$ S_0 \leq 4 \cdot (|C_{11}|^2 + |C_{22}|^2) \left( 1 - \frac{|C_{12}|^2 |C_{21}|^2}{|C_{22}|^2 |C_{11}|^2} \right) \cdot (|C_{11}|^2 + |C_{22}|^2 + |C_{21}|^2 \left( 2 - 3\frac{|C_{12}|^2}{|C_{22}|^2} \right) + |C_{12}|^2 \left( 2 - 3\frac{|C_{21}|^2}{|C_{11}|^2} \right))^{-1} $$

$$ \Delta S_0 \leq (|C_{21}|^2 + |C_{22}|^2) \left( 1 - \frac{|C_{12}|^2 |C_{21}|^2}{|C_{22}|^2 |C_{11}|^2} \right) \cdot (|C_{11}|^2 + |C_{22}|^2 + |C_{21}|^2 \left( 2 - 3\frac{|C_{12}|^2}{|C_{22}|^2} \right) + |C_{12}|^2 \left( 2 - 3\frac{|C_{21}|^2}{|C_{11}|^2} \right))^{-1} $$

independent of delay.

**Proof:** (7) gives

$$ |C_{21}|^2 P_1 + |C_{22}|^2 P_2 \geq 2^{R_2} \left( \frac{|C_{21}|^2}{|C_{11}|^2} 2^{R_1} - \frac{|C_{21}|^2}{|C_{11}|^2} + 1 \right) - 1 \quad (9) $$

$$ |C_{11}|^2 P_1 + |C_{12}|^2 P_2 \geq 2^{R_1} \left( \frac{|C_{12}|^2}{|C_{22}|^2} 2^{R_2} - \frac{|C_{12}|^2}{|C_{22}|^2} + 1 \right) - 1 \quad (10) $$

In the equal rate scenario, $R_1, R_2 = \frac{R_s}{2}$, and our objective is to minimize the sum power $P_1 + P_2$. We construct following optimization problem

$$ \min \quad P_1 + P_2 \quad \text{s.t.} \quad A \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} \geq b $$

where $A = \begin{pmatrix} |C_{21}|^2 & |C_{22}|^2 \\ |C_{11}|^2 & |C_{12}|^2 \end{pmatrix}$ and $b = \begin{pmatrix} 2^{R_2/2} \left( \frac{|C_{21}|^2}{|C_{11}|^2} 2^{R_1/2} - \frac{|C_{21}|^2}{|C_{11}|^2} + 1 \right) - 1 \\ 2^{R_1/2} \left( \frac{|C_{12}|^2}{|C_{22}|^2} 2^{R_2/2} - \frac{|C_{12}|^2}{|C_{22}|^2} + 1 \right) - 1 \end{pmatrix}$. Using simple linear programming principles, one optimal solution can be found at the vertex of the feasible region. That is, $P_1 + P_2|_{\text{min}} = P_{1o} + P_{2o}$ where $\begin{pmatrix} P_{1o} \\ P_{2o} \end{pmatrix} = A^{-1} b > 0$. We solve this simple linear system and get

$$ P_{1o} = |C_{11}|^{-2} \cdot 2^{R_2} \left( \frac{2^{R_2} - 1}{|C_{21}|^2 |C_{22}|^2} \left( 1 - \frac{|C_{12}|^2}{|C_{22}|^2} \right) + \left( 1 - \frac{|C_{21}|^2}{|C_{11}|^2} \right) \frac{2^{R_2} - 1}{|C_{21}|^2 |C_{22}|^2} \right) $$

$$ P_{2o} = |C_{22}|^{-2} \cdot 2^{R_2} \left( \frac{2^{R_2} - 1}{|C_{21}|^2 |C_{22}|^2} \left( 1 - \frac{|C_{12}|^2}{|C_{22}|^2} \right) + \left( 1 - \frac{|C_{21}|^2}{|C_{11}|^2} \right) \frac{2^{R_2} - 1}{|C_{21}|^2 |C_{22}|^2} \right) $$

Now we have the expression of sum power as a function of sum rate. Follow similar technique as (140)–(144) in [22], it is easy to show that

$$ \frac{E_b}{N_0 \min} = \left. \frac{d \text{SNR}(R)}{dR} \right|_{R=0} \quad \text{and} \quad S_0 = \left. \frac{d^2 \text{SNR}(R)}{dR^2} \right|_{R=0} \log 2 \quad (13) $$
Combining (11), (12) and (13), we have

\[
\frac{E_b}{N_0 \min} = \left(\frac{|C_{11}|^2 + |C_{22}|^2}{2}\right) \log_2 2
\]

\[
S_0 = 4 \cdot (|C_{11}|^2 + |C_{22}|^2) \left(1 - \frac{|C_{12}|^2}{|C_{22}|^2 |C_{11}|^2}\right) \cdot (|C_{11}|^2 + |C_{22}|^2)
\]

\[
+ |C_{21}|^2 \left(2 - 3 \frac{|C_{12}|^2 |C_{11}|^2}{|C_{22}|^2}\right) + |C_{12}|^2 \left(2 - 3 \frac{|C_{21}|^2 |C_{11}|^2}{|C_{22}|^2}\right)\right)^{-1}
\]

We also have

**Theorem 4.** The outer bound stated in Theorem 1 [3], which characterizes the Shannon sum capacity of a 2-user scalar Gaussian interference channel without delay in the low interference region, also outer bounds the capacity region of a channel with delay defined by (6).

**Proof:** The outer bound developed by Theorem 1 [3] has two parts. Part one is composed of (2[3]) and (3[3]), which is identical to Kramer’s bound. Part two is a genie-aided bound, expressed by (1[3]). For Kramer’s bound, we have already shown that it is adaptable to the delay model. For the genie-aided part, we only briefly sketch how the proof in [3] needs to be modified, limiting it to the case of a channel with real fading coefficients and unit direct-link gains. The generalization of this bound into non-canonical complex channel can be done as in [24].

The channel model is

\[
Y_1^n = X_1^n + \sqrt{a} \tilde{X}_2^n + Z_1^n
\]

\[
Y_2^n = \sqrt{b} \tilde{X}_1^n + X_2^n + Z_2^n
\]

(14)

where length \( n \) vectors \( Y_j^n = (y_j[1], \ldots, y_j[n]) \), \( X_j^n = (x_j[0], \ldots, x_j[n-1]) \), and \( \tilde{X}_j^n = (\tilde{x}_j[0], \ldots, \tilde{x}_j[n-1]) \). Following [3], we provide genie signals \( S_1^n \) and \( S_2^n \) to receiver 1 and 2 respectively, where both are noisy version of the receivers’ desired signals:

\[
S_1^n = X_1^n + W_1^n
\]

\[
S_2^n = X_2^n + W_2^n
\]

where \( W_1^n \) and \( Z_2^n \) are zero mean, unit variance IID Gaussian noise. The correlation between \( W_j^n \) and \( Z_j^n \) is \( \rho_{jj} \). Similar to equation (9) in [3] we can write

\[
n \left( R_1 + \mu R_2 \right) \\
\leq I \left( X_1^n; Y_1^n \right) + \mu I \left( X_2^n; Y_2^n \right) + o(n) \\
\leq I \left( X_1^n; Y_1^n, S_1^n \right) + \mu I \left( X_2^n; Y_2^n, S_2^n \right) + o(n) \\
= h \left( Y_1^n | X_1^n; W_1^n \right) + \mu h \left( Y_2^n | X_2^n; W_2^n \right) \\
\quad + h \left( X_1^n + W_1^n - \mu h \left( \sqrt{a} \tilde{X}_2^n + Z_2^n | W_2^n \right) \right) \\
\quad + h \left( X_2^n + W_2^n - \mu h \left( \sqrt{b} \tilde{X}_1^n + Z_1^n | W_1^n \right) \right) \\
\quad - \mu \left( h \left( W_1^n \right) + h \left( W_2^n \right) \right) + o(n) \\
\overset{(a)}{=} h \left( Y_1^n | X_1^n + W_1^n \right) + \mu h \left( Y_2^n | X_2^n + W_2^n \right) \\
\quad + h \left( X_1^n + W_1^n - \mu h \left( \sqrt{a} \tilde{X}_2^n + Z_2^n | W_2^n \right) \right) \\
\quad + h \left( X_2^n + W_2^n - \mu h \left( \sqrt{b} \tilde{X}_1^n + Z_1^n | W_1^n \right) \right) \\
\quad - \mu \left( h \left( W_1^n \right) + h \left( W_2^n \right) \right) + o(n) \\
\overset{(b)}{\leq} nh \left( Y_{1G} | X_{1G} + W_1 \right) + \mu h \left( Y_{2G} | X_{2G} + W_2 \right) \\
\quad + h \left( X_1^n + W_1^n - \mu h \left( \sqrt{a} \tilde{X}_2^n + Z_2^n | W_2^n \right) \right) \\
\quad + h \left( X_2^n + W_2^n - \mu h \left( \sqrt{b} \tilde{X}_1^n + Z_1^n | W_1^n \right) \right) \\
\quad - \mu \left( h \left( W_1^n \right) + h \left( W_2^n \right) \right) + o(n) \\
\]

(15)

\(^2\)See also [23], [1], [2]
where \( \lim_{n \to \infty} o(n)/n = 0 \) and
\[
\hat{Y}_1^n = X_1^n + \sqrt{a}X_2^n + Z_1^n \\
\hat{Y}_2^n = \sqrt{b}X_1^n + X_2^n + Z_2^n
\]
Equality \((a)\) holds because both \( X_j \) and \( \hat{X}_j \) can be obtained from sampling the same continuous-time base band signal \( X_j(t) \) at the Nyquist rate, while \( Z_j \) and \( W_j \) are sampled from white Gaussian noise, so that
\[
h \left( \sqrt{b}X_1^n + Z_2^n | W_2^n \right) = h \left( \sqrt{b}\hat{X}_1^n + Z_2^n | W_2^n \right) + o(n)
\]
\[
h \left( \sqrt{a}X_2^n + Z_1^n | W_1^n \right) = h \left( \sqrt{a}\hat{X}_2^n + Z_1^n | W_1^n \right) + o(n)
\]
Inequality \((b)\) holds if the following inequality is true
\[
h \left( Y_1^n | X_1^n + W_1^n \right) \overset{(c)}{\leq} nh \left( Y_{1G} | X_{1G} + W_1 \right)
\]
\[
h \left( \hat{Y}_{1G} | X_{1G} + W_1 \right) \overset{(d)}{\leq} nh \left( \hat{Y}_{1G} | X_{1G} + W_1 \right)
\]
Given Lemma 1[1]. \((c)\) holds, and equality is achieved iff \( X_1^n \) and \( \hat{X}_1^n \) are IID zero-mean Gaussian, satisfying power constraints \( E[|X_1^2|] \leq P_1, E[|\hat{X}_1^2|] \leq \bar{P}_2 \). \((d)\) is true because time-shifting of a signal sampled at the Nyquist rate does not change signal power, so \( \bar{P}_2 \leq \bar{P}_2 \), and equality holds iff \( X_1^n \) is IID zero-mean Gaussian satisfying power constraints \( E[|X_2^2|] \leq P_2 \). This shows that the sum capacity of a channel with delay is outer bounded by that of a channel without delay.

**Corollary 5.** For the channel defined by (6), \( K = 2 \), if the channel coefficients satisfy
\[
\sqrt{\frac{|C_{12}|^2}{|C_{22}|^2}} + \sqrt{\frac{|C_{21}|^2}{|C_{11}|^2}} \leq 1
\]
then in the equal power case, the optimal sum slope \( S_0 \) can be achieved by letting each transmitter use IID Gaussian signaling, and each receiver treat interference as noise giving
\[
S_0 = 2 \left( \frac{|C_{11}|^2 + |C_{22}|^2}{|C_{11}|^2 + |C_{22}|^2} \right)^2 \quad (16)
\]
\[
\Delta S_0 = 1 + 2 \left( \frac{|C_{11}|^2 |C_{12}|^2 + |C_{21}|^2 |C_{22}|^2}{|C_{11}|^2 + |C_{22}|^2} \right)^2 \quad (17)
\]

**Proof:** Following Proposition 7 in [24], a sufficient condition for the optimality of treating interference as noise in the 2-user interference channel is
\[
\sqrt{\frac{|C_{12}|^2}{|C_{22}|^2}} \left( 1 + |C_{21}|^2 \text{SNR}_1 \right) + \sqrt{\frac{|C_{21}|^2}{|C_{11}|^2}} \left( 1 + |C_{12}|^2 \text{SNR}_2 \right) \leq 1
\]
Under both criterion 1 and 2 where \( \text{SNR}_i \) approaches zero, the noisy interference condition above becomes
\[
\sqrt{\frac{|C_{12}|^2}{|C_{22}|^2}} + \sqrt{\frac{|C_{21}|^2}{|C_{11}|^2}} \leq 1
\]
The sum rate achieved by treating interference as noise is
\[
R_s \leq \log \left( 1 + \frac{|C_{11}|^2 \text{SNR}_2}{1 + |C_{12}|^2 \text{SNR}_2} \right) + \log \left( 1 + \frac{|C_{22}|^2 \text{SNR}_2}{1 + |C_{21}|^2 \text{SNR}_2} \right)
\]

Figure 1 illustrates the sum slope region of a 2-user interference channel with unit direct link gain, and symmetric cross link gain, that is, \( |C_{11}|^2 = |C_{22}|^2 = 1 \), and \( |C_{12}|^2 = |C_{21}|^2 = a \). Notice that under this assumption, equal power and equal rate scenarios are equivalent. For weak interference, when \( \alpha \leq \frac{1}{2} \), the best known inner bound is achieved by treating interference as noise (TIN), \( S_{TIN} = \frac{4}{1 + 2\alpha} \), when \( \frac{1}{2} < \alpha \leq 1 \), the best known inner bound is achieved by TDMA, \( S_{TDMA} = 2 \). Further, if \( \alpha \leq \frac{1}{4} \), this is a noisy interference channel, where treating interference as noise achieves optimal sum slope. The best outer
bound for the weak interference channel is derived from Kramer’s capacity bound, $S_{outer} = \frac{4(1+a)}{1+3a}$. When $a > 1$, the sum slope is equal to 4, equal to that of a 2-user channel with no interference.

The focal point here is the point $a = 1$. Just above that, the effect of interference is completely eliminated. Just below that, interference is at its worst. One could wonder if, for the $K$-user case, the former fact could be used effectively. It turns out that is not the case. In fact, the $K$-user interference channel mostly operates at the point just below $a = 1$.

![Figure 1: Sum rate wideband slope versus $\frac{|C_{21}|}{|C_{11}|}$](image)

**IV. THE $K$-USER CASE**

The $K$-user case is very different from the two user case, just as for the high SNR case considered in [4]. As for the high SNR case we can obtain a significant increase in rate by using a variation of interference alignment. Specifically, we let users transmit in even time symbols, and we align interference in the odd time symbols using delay differences, as in [16], [17]. However, delay is much more efficient when we let $B \to \infty$ as $n_{ij} = \lceil \tau_{ij} \frac{B}{2} \rceil$ can become arbitrarily large. We therefore do not need to use approximations as in [16] or large $K$ as in [17].

We first need to refine the definition of wideband slope as

$$\frac{E_b}{N_0 \min} = \lim \inf_{B \to +\infty} \frac{P}{R \cdot N_0 B}$$

$$S_0 \triangleq \lim \sup_{\frac{E_b}{N_0} \to \frac{E_b}{N_0 \min}} \frac{C \left( \frac{E_b}{N_0} \right)}{10 \log_{10} \frac{E_b}{N_0} - 10 \log_{10} \frac{E_b}{N_0 \min}} 10 \log_{10} 2$$

For comparison purposes, we will list the results for the interference-free case, i.e., $C_{ij} = 0$ for $i \neq j$ as follows (directly obtained from [13, Theorem 9]):

$$S_{0,\text{no interference}} = 2 \left( \sum_j |C_{jj}|^2 \right)^{\frac{1}{2}}$$  

for equal power

$$S_{0,\text{no interference}} = 2K$$  

for equal rate

**A. Achievable rate**

The proof of the achievable rate is based on the following result

**Lemma 6.** If $\tau_{ji}$ are linearly independent over the rational numbers, then for any $\delta > 0$, there exist arbitrarily large real numbers $B$ and integers $n_{ji}$, such that

$$|\tau_{ji} B - 2n_{ji} - 1| \leq \delta$$

The proof of Lemma 6 is based on the following fundamental approximation results in number theory.
**Theorem 7.** ([25], Theorem 7.9) (First form of Kronecker’s theorem) If \( \alpha_1, \ldots, \alpha_n \) are arbitrary real numbers, if \( \theta_1, \ldots, \theta_n \) are linearly independent real numbers over the rational numbers, and if \( \epsilon > 0 \) is arbitrary, then there exists a real number \( t \) and integers \( h_1, \ldots, h_n \) such that

\[
|t \theta_i - h_i - \alpha_i| < \epsilon
\]

for \( i = 1, 2, \ldots, n \).

And

**Lemma 8.** ([25], Exercise 7.7, p. 160) Under the hypotheses of Theorem 7, if \( T > 0 \) is given, there exists a real \( t > T \) satisfying the \( n \) inequalities (21).

Proof of Lemma 6: Let \( \alpha_1, \ldots, \alpha_n = 0.5 \), \( \epsilon = \frac{\delta}{2} \). According to Theorem 7, there exist arbitrarily large real number \( \hat{B} \) and an integer \( n_j \), such that

\[
|\tau_{ji} \hat{B} - n_j - 0.5| \leq \frac{\delta}{2}
\]

Now let \( B = 2 \hat{B} \), we have

\[
|\tau_{ji}B - 2n_j - 1| \leq \delta
\]

Combining with Lemma 8, Lemma 6 is proved.

**Lemma 9.** Under the assumptions \( x_j [2m] \) are IID Gaussian \( \sim \mathcal{N} (0, 2P_j) \) and \( x_j [2m + 1] = 0 \) for all \( j \) and \( m \), \( E [\hat{x}_i [n_1, \delta_{ji}] \hat{x}_i [n_2, \delta_{ji}]] \) is a continuous function of \( \delta_{ji} \), and

\[
\lim_{\delta_{ji} \to 0} E [\hat{x}_i^* [n_1, \delta_{ji}] \hat{x}_i [n_2, \delta_{ji}]] = \begin{cases} 2P_i & \text{if } n_1 = n_2 = 2k, \\ 0 & \text{for some integer } k, \\ \text{o.w.} & \end{cases}
\]

Proof: We have

\[
E [\hat{x}_i^* [n_1, \delta_{ji}] \hat{x}_i [n_2, \delta_{ji}]] = E \left[ \left( \sum_{m=-\infty}^{\infty} x_i [2m] \text{sinc}(n_1 - 2m + \delta_{ji}) \right) \right. \\
\left. \left( \sum_{m=-\infty}^{\infty} x_i [2m] \text{sinc}(n_2 - 2m + \delta_{ji}) \right) \right] \\
= \sum_{m=-\infty}^{\infty} \left( E \left[ \left| x_i [2m] \right|^2 \right] \text{sinc}(n_1 - 2m + \delta_{ji}) \text{sinc}(n_2 - 2m + \delta_{ji}) \right) \\
= \sum_{m=0}^{\infty} \left( E \left[ \left| x_i [2m] \right|^2 \right] \text{sinc}(n_1 - 2m + \delta_{ji}) \text{sinc}(n_2 - 2m + \delta_{ji}) \right) \\
+ \sum_{m=1}^{\infty} \left( E \left[ \left| x_i [-2m] \right|^2 \right] \text{sinc}(n_1 + 2m + \delta_{ji}) \text{sinc}(n_2 + 2m + \delta_{ji}) \right) \\
= 2P_i \left( \sum_{m=0}^{\infty} \text{sinc}(n_1 - 2m + \delta_{ji}) \text{sinc}(n_2 - 2m + \delta_{ji}) \right) \\
+ \sum_{m=1}^{\infty} \text{sinc}(n_1 + 2m + \delta_{ji}) \text{sinc}(n_2 + 2m + \delta_{ji}) \right)
\]

Define \( f_m (\delta_{ji}) \) and \( g_m (\delta_{ji}) \) as

\[
f_m (\delta_{ji}) \triangleq \text{sinc}(n_1 - 2m + \delta_{ji}) \text{sinc}(n_2 - 2m + \delta_{ji}) \\
g_m (\delta_{ji}) \triangleq \text{sinc}(n_1 + 2m + \delta_{ji}) \text{sinc}(n_2 + 2m + \delta_{ji})
\]

and their partial sums \( s_{f,M} (\delta_{ji}) = \sum_{m=0}^{M} f_m (\delta_{ji}) \), \( s_f (\delta_{ji}) = \lim_{M \to \infty} s_{f,M} (\delta_{ji}) \); \( s_{g,M} (\delta_{ji}) = \sum_{m=0}^{M} g_m (\delta_{ji}) \), \( s_g (\delta_{ji}) = \lim_{M \to \infty} s_{g,M} (\delta_{ji}) \). Here

\[
|f_m (\delta_{ji})| = \left| \frac{\sin (\pi(n_1 - 2m + \delta_{ij})) \sin (\pi(n_2 - 2m + \delta_{ij}))}{(n_1 - 2m + \delta_{ij})(n_2 - 2m + \delta_{ij})} \right| \leq \frac{1}{(n_1 - 2m + \delta_{ij})(n_2 - 2m + \delta_{ij})}
\]
and
\[ |g_m(\delta_{ji})| \leq \frac{1}{(n_1 + 2m + \delta_{ij})(n_2 + 2m + \delta_{ij})} \]

Let \( M_{f,k} \triangleq \frac{1}{(n_1 - 2m + \delta_{ij})(n_2 - 2m + \delta_{ij})} \), \( M_{g,k} \triangleq \frac{1}{(n_1 + 2m + \delta_{ij})(n_2 + 2m + \delta_{ij})} \). Because \( \sum_{m=1}^{\infty} \frac{1}{k} \) is convergent, \( \sum_{k=0}^{\infty} M_{f,k} \) and \( \sum_{k=1}^{\infty} M_{g,k} \) converges too. Due to Weierstrass’s test for uniform convergence[26], \( s_{f,M}(\delta_{ji}) \) and \( s_{g,M}(\delta_{ji}) \) converge uniformly. And using Theorem 7.11 in [26], we have
\[
\lim_{\delta_{ji} \downarrow 0} \lim_{M \to \infty} s_{f,M}(\delta_{ji}) = \lim_{M \to \infty} \lim_{\delta_{ji} \downarrow 0} s_{f,M}(\delta_{ji})
\]
\[
\lim_{\delta_{ji} \downarrow 0} \lim_{M \to \infty} s_{g,M}(\delta_{ji}) = \lim_{M \to \infty} \lim_{\delta_{ji} \downarrow 0} s_{g,M}(\delta_{ji})
\]
Thus, (22) becomes
\[
\lim_{\delta_{ji} \downarrow 0} E\left[\hat{x}_i^*[n_1,\delta_{ji}][\hat{x}_i[n_2,\delta_{ji}]]\right] = 2P_1\left(\lim_{M \to \infty} \lim_{\delta_{ji} \downarrow 0} s_{f,M}(\delta_{ji}) \right. \\
+ \left. \lim_{M \to \infty} \lim_{\delta_{ji} \downarrow 0} s_{g,M}(\delta_{ji}) \right) = \begin{cases} 2P_1 & \text{if } n_1 = n_2 = 2k, \\
0 & \text{o.w.} \end{cases}
\]
And given Theorem 7.12 in [26] and the continuity of sinc function, we can conclude that \( E[\hat{x}_i[n_1,\delta_{ji}][\hat{x}_i[n_2,\delta_{ji}]] \) is a continuous function of \( \delta_{ji} \).

**Theorem 10.** Suppose that the set of delays \( \{\tau_{ij}\} \) are linearly independent over the rational numbers. Then the following wideband slope is achievable
\[
S_0 = \frac{\left(\sum_{j} |C_{jj}|^2\right)^2}{\sum_{j} |C_{jj}|^4} \quad \text{equal power}
\]
\[
S_0 = K \frac{\sum_{j} |C_{jj}|^4}{|C_{jj}|^2} \quad \text{equal rate}
\]
In both cases
\[
\Delta S_0 = \frac{1}{2}
\]
is achievable.

**Proof:** Transmitter \( j \) uses a standard i.i.d Gaussian code book of power \( 2P_j \), and transmits this code book using even time symbols so that \( x_j[2n+1] = 0 \); the transmitted power therefore is \( P_j \). Receiver \( j \) uses only the even time symbols for decoding and ignores the odd time symbols. Because of Lemma 6 we can achieve that the interference is confined at odd time symbols. However, due to the fractional delays \( \delta_{ij} \) (3) there is a leakage of interference to even time symbols. This interference is treated as noise. Let
\[
\epsilon_j(B) = \sum_{i \neq j} K \left| E\left[(\hat{x}_i[2n])^2\right]\right|
\]
denote the power of the leaked interference.

The best rate with this scheme is clearly achieved if the leaked interference power is zero; in that case the channel is an interference-free channel where half the symbols are not used. We can therefore conclude
\[
\Delta S_0 \leq \frac{1}{2}
\]
(23)

On the other hand, taking into account the leaked interference, the achievable rate at receiver \( j \) is
\[
R_j = \frac{1}{2} \log \left(1 + \frac{|C_{jj}|^2 P_j^2}{|C_{jj}|^2 + \epsilon_j(B)} \right) \left(\frac{1}{BN_0}\right)^2 + \frac{1}{BN_0} \right)
\]
(24)
\[
= \frac{|C_{jj}|^2 P_j}{BN_0} - \left(\epsilon_j(B) |C_{jj}|^2 P_j + |C_{jj}|^4 P_j^2 \right) \left(\frac{1}{BN_0}\right)^2 + o\left(\frac{1}{BN_0} \right)^2 \right)
\]
(25)
The wideband slope is a continuous function of the coefficients in the first two terms in the Taylor series of \( R_j \) in \( 1/B \). According to Lemma 6 for any \( \delta > 0 \) there exists some \( B_\delta \) so that for \( B > B_\delta \) the delay \( n_{ij} = \lceil \tau_{ij} B + \frac{1}{2} \rceil \) is odd and
\[
|\delta_{ij}| = |\tau_{ij} B - \lceil \tau_{ij} B + \frac{1}{2} \rceil| \leq \delta \quad i \neq j
\]

From Lemma 9 we can then conclude that there exists a sequence \( \{ B_k, k = 1 \ldots \infty \} \) so that \( B_k \to \infty \) and \( \epsilon_j(B_k) \to 0 \), \( j = 1 \ldots K \). This means that \( \Delta S_0 = 1/2 \) is a limit point, and together with (23) this shows that \( \Delta S_0 = 1/2 \) is the limit superior.

We can easily conclude

**Corollary 11.** Suppose that node positions have independent positions and each node position is a continuous distribution. Then

\[
\Delta S_0 = 1/2
\]

is achievable.

So in practice \( \Delta S_0 = 1/2 \) is achievable, since nodes can never be positioned accurately in a grid; there is always some nano-scale inaccuracy (dither) in positions, at the fundamental level due to quantum mechanics.

### B. Practical Implementation

In this section we will show that the interference alignment ideas of the previous section can be used in a practical system, and show some simulation results. This will also make it clear why the modified definition (19) is needed.

We use the following transmission scheme

1) **Encoding**
   a) Transmitters \( j \) pick their messages with uniform probability over index sets \( (1, 2, \ldots, 2^{mR_j}) \), then encode the messages into length \( m \) codewords \( x_j^m \) using codebooks \( (2^{mR_j}, m) \). The code books are generated by IID Gaussian distribution \( \mathcal{N}(0, 2^{P_j}) \). \( R_j \) are the spectral efficiency of user \( j \).
   b) Extend \( x_j^m \) into length \( n = 2m \) sequences \( x_j^n \) using the following rule:
   \[
x_j[n] = \begin{cases} 
   \hat{x}_j[k] & \text{if } n = 2k \\
   0 & \text{if } n = 2k + 1
   \end{cases}
   \]
   c) Construct the continuous-time base band input \( x_j(t) \) by sinc interpolation
   \[
x_j(t) = \sum_{n=-\infty}^{\infty} x_j[n] \text{sinc}(t-n)
   \]

2) **Decoding**
   Assume that the received baseband signals are sampled symbol-synchronously with the desired signal. We recover \( \hat{x}_j^m \) from \( y \left( \frac{2k}{B} \right) \) using typical set decoding, treating leaked interference from the odd-time symbols as interference.

In the simulation, we consider a 3-user channel with symmetric channel gain: \( |C_{ij}|^2 = 1, |C_{ji}|^2 = 0.8 \). Notice that for channels with symmetric link gains, equal power and equal rate scenarios are equivalent. The delays \( \tau_{ij} \) are chosen such that they are linearly independent over the rational numbers. We assume \( m \) to be infinite, then the achievable Shannon rate for user \( j \) is \( R_j \), as that stated in (24). We are interested in the spectral efficiency of one channel realization under a sequence of large but finite \( B \) values. Simulation results are given and analysed below.

Fig. 2a shows the simulation results of the case where bandwidth \( B \) increases continuously. The system performance shows a noticeable oscillating behavior, which is the main reason why we need the modified definition (19). This phenomenon can be explained as followed. At receiver \( j \) the interference caused by user \( i \) is an increasing function of \( \delta_{ji} \); and \( \delta_{ji} \) is a periodic function of \( B \), oscillating between 0 and 1. It can be proved that the cumulative effect of leaked interferences from all other users has same (almost) periodic behavior. The proof is similar to that of Lemmas 6 and 8; details are skipped here. As a result, while there exists an infinite sequence \( B = (B_1, B_2, \ldots) \) that can make the leaked sum interference arbitrarily small, there also exists some infinite sequence of \( B \) values under which a large proportion of interference energy leaks onto signal space, resulting in poor performance.

From the discussion above, we can see that the choice of operating bandwidth (really, sampling frequency) relative to delays has a large influence on practical performance. Define the \( B_\delta \) as \( \{ B_\delta : |\tau_{ji} B_\delta - 2n_{ji} - 1| \leq \delta \forall i, j \} \). Lemma 6 shows that an infinite sequence \( B_\delta = (B_1^\delta, B_2^\delta, \ldots) \) exists for arbitrarily small \( \delta > 0 \). Theorem 7.10 in [25] provides the Second form of Kronecker’s theorem, which shows that if the real number \( t \) replaced by integer \( k \), the results in Theorem 7 and Lemma 8 still hold. As a result, we have the following Lemma, which is a second form of Lemma 6.
Lemma 12. If \( \tau_{ji} \) are linearly independent over the rational numbers, then for any \( \delta > 0 \), there exist arbitrarily large integers \( B \) and integers \( n_{ji} \), such that

\[
|\tau_{ji}B - 2n_{ji} - 1| \leq \delta
\]

Lemma 12 says that there exists some \( B_\delta \) which is a subsequence of \( B = (1, 2, 3, \cdots) \). Thus, the brute force algorithm of searching through all integer \( B \) is guaranteed to find good operating bandwidth values. Fig. 2b shows the performance of the proposed achievable scheme when the system operating at a sequence of \( B_\delta, \delta = 0.2 \). However, designing more efficient \( B_\delta \)-searching algorithm is still an open problem.

![Simulation Results](image)

Figure 2: Achievable spectral efficiency versus \( \frac{E_b}{N_0} \). The straight line shows the performance approximated to first order by the wideband slope.

C. Outer bounds

In Section IV-A we have seen that \( \Delta S_0 = \frac{1}{2} \) can be achieved. Is this the best possible? Clearly no. In the two user channel the interference alignment scheme reduces to TDMA, and we have seen in Section III that interference decoding and treating interference as noise can be better than TDMA. And it is not difficult to construct examples with more users where \( \Delta S_0 > \frac{1}{2} \) is achievable. However, we will show that for large \( K \) this happens rarely. The outer bounds in this section are proven for all channel coefficients \( C_{ij} \) IID. However, this is not a necessary condition, only a convenient condition to simplify proofs; later in the section we will comment more on this.

We say that users \((i, j)\) form an \((1 - \epsilon)\)-interference pair if

\[
1 - \epsilon \leq \frac{|C_{ji}|^2}{|C_{ii}|^2}, \quad \frac{|C_{ij}|^2}{|C_{jj}|^2} < 1
\]

and a weak \((1 - \epsilon)\)-interference pair if

\[
1 - \epsilon \leq \frac{|C_{ji}|^2}{|C_{ii}|^2} < 1 \quad \text{or} \quad 1 - \epsilon \leq \frac{|C_{ij}|^2}{|C_{jj}|^2} < 1
\]

First, we will consider the equal rate case. We assume that the channel coefficients \( C_{ij} \) are IID and \( E[|C_{ii}|^{-2}] < \infty \); if the latter assumption were not satisfied, \( \lim_{K \to \infty} \frac{1}{K} \sum_{i=1}^{K} P_i = \infty \) even in the interference-free case, so the energy per bit and wideband slope would not be well-defined for large \( K \). Let \( F_{|C_{ii}|^2} \) be the CDF of \( |C_{ii}|^2 \); this defines a measure on the real numbers through \( u_F((a, b)) = F_{|C_{ii}|^2}(b) - F_{|C_{ii}|^2}(a) \). For \( \forall \epsilon > 0 \), we define the following sets

\[
R_\epsilon = \{ x \in \mathbb{R} : F_{|C_{ii}|^2}(x) - F_{|C_{ii}|^2}((1 - \epsilon) x) < \epsilon \}
\]

\[
D_\epsilon = \{ i \in \{1, \ldots, K\} : |C_{ii}|^2 \in R_\epsilon \}
\]

\[
B_{\epsilon, \hat{\epsilon}} = \{ i \in D_\epsilon : \text{user does not form an } (1 - \epsilon) \text{-interference pair with any other user} \} \]
Lemma 13. For all $\epsilon, \hat{\epsilon} > 0$,

$$\lim_{K \to \infty} \Pr (B_{\epsilon, \hat{\epsilon}} = \emptyset) = 1$$

Lemma 14. Given any infinite sequence $\hat{\epsilon}_n > 0$ satisfying $\hat{\epsilon}_n > \epsilon_{n+1}$ and $\hat{\epsilon}_n \to 0$, the corresponding sequence of $R_{\epsilon_n}$ satisfies

1) $R_{\hat{\epsilon}_{n+1}} \subseteq R_{\epsilon_n}$;
2) $\mu_F (R_{\epsilon_n}) \to 0$.

Lemma 15. Let $G_i$ be a sequence of events with $G_{i+1} \subseteq G_i$ and $\lim_{i \to \infty} \mu (G_i) = 0$. Let $X$ be a positive random variable with $E[X] < \infty$. Define

$$X_i = \begin{cases} X & X \in G_i \\ 0 & X \notin G_i \end{cases}$$

Then

$$\lim_{i \to \infty} E[X_i] = 0$$

The proofs of the Lemmas are in the Appendix.

Theorem 16. Suppose that the channel coefficient $C_{ij}$ are IID. In the equal rate case

$$\forall \delta > 0 : \lim_{K \to \infty} \Pr \left( \Delta S_0 \leq \frac{1}{2} + \delta \right) = 1$$

Proof: We discuss users in set $D_\bar{\epsilon}$ and those in set $D_\epsilon$ separately.

First, let us look at user $j$, $j \in D_\bar{\epsilon}$. We assume that each user $j \in D_\bar{\epsilon}$ forms a $(1 - \epsilon)$-interference pair with some user $i(j)$. Given Lemma 13, this happens with high probability. Consider a single $(1 - \epsilon)$-interference pair $(j, i(j))$. We can get an upper bound on the spectral efficiency, by eliminating all interference links except the links between users $j$ and $i(j)$, so that the received signal is

$$y_j = C_{jj} x_j + C_{ji(j)} x_{i(j)} + z_j$$
$$y_{i(j)} = C_{i(j)j} x_j + C_{i(j)i(j)} x_{i(j)} + z_{i(j)}$$

Let

$$\frac{|C_{jj}|^2}{|C_{ji(j)}|^2} = 1 - \epsilon_{ji(j)}, \frac{|C_{i(j)j}|^2}{|C_{ji(j)}|^2} = 1 - \epsilon_{i(j)j}$$

Since $\{y_j, y_{i(j)}\}$ is a $(1 - \epsilon)$-interference pair, we have

$$0 \leq \epsilon_{ji(j)}, \epsilon_{i(j)j} < \epsilon$$

Applying (11) and (12), to $\{y_j, y_{i(j)}\}$, we have the optimum solution

$$P_{i(j)o} = |C_{i(j)i(j)}|^{-2} \cdot \frac{\frac{2^{R_S}}{2} - 1 \left(1 - \epsilon_{i(j)j}\right) \epsilon_{ji(j)} + \epsilon_{i(j)j} \left(\frac{2^{R_S}}{2} - 1\right)}{1 - \left(1 - \epsilon_{ji(j)}\right) \left(1 - \epsilon_{i(j)j}\right)}$$

$$P_{jo} = |C_{jj}|^{-2} \cdot \frac{\frac{2^{R_S}}{2} - 1 \left(1 - \epsilon_{ji(j)}\right) \epsilon_{i(j)j} + \epsilon_{i(j)j} \left(\frac{2^{R_S}}{2} - 1\right)}{1 - \left(1 - \epsilon_{ji(j)}\right) \left(1 - \epsilon_{i(j)j}\right)}$$

And $P_{i(j)} + P_j \geq P_{i(j)o} + P_{jo}$. Notice that the RHS of (31) and (32) are monotonically decreasing function of either $\epsilon_{ji(j)}$ or $\epsilon_{i(j)j}$. Thus, given the condition (28), we can relax (31) and (32) by substituting $\epsilon_{ji(j)}$ and $\epsilon_{i(j)j}$ by $\epsilon$,

$$P_{i(j)o} \geq |C_{i(j)i(j)}|^{-2} \cdot \frac{\frac{2^{R_S}}{2} \left(1 - \epsilon\right) \frac{2^{R_S}}{2} + \epsilon - 1}{2 - \epsilon}$$

$$P_{jo} \geq |C_{jj}|^{-2} \cdot \frac{\frac{2^{R_S}}{2} \left(1 - \epsilon\right) \frac{2^{R_S}}{2} + \epsilon - 1}{2 - \epsilon}$$

Thus,

$$P_{jo} = \frac{\frac{2^{R_S}}{2} \left(1 - \epsilon\right) \frac{2^{R_S}}{2} + \epsilon}{2 - \epsilon} |C_{jj}|^{-2}, \text{ if } j \in D_\epsilon$$

Second, for user $k, k \in D_\bar{\epsilon}$, we treat them as being interference-free. In this case, we have

$$P_k \geq \left(\frac{2^{R_S}}{2} - 1\right) |C_{kk}|^{-2}, \text{ if } k \in D_\bar{\epsilon}$$
Combining (32) and (34), the minimum sum power required for an equal rate system with sum rate $R_s$ is lower bounded by

$$P_s \geq \frac{2^{\frac{2\pi}{\theta}} (1 - \epsilon) 2^{\frac{2\pi}{\theta} + \epsilon}}{2 - \epsilon} \sum_{j \in D^c} |C_{jj}|^{-2}$$

$$+ \left(2^{\frac{2\pi}{\theta}} - 1 \right) \sum_{k \in D_s} |C_{kk}|^{-2}$$

Using (13) on (35) we get

$$\frac{E_b}{N_0 \min} = \frac{\sum |C_{jj}|^{-2}}{K} \log 2$$

$$\Delta S_0 = \frac{(2 - \epsilon)}{(4 - 3\epsilon)(1 - \theta) + (2 - \epsilon) \theta}$$

where

$$\theta \triangleq \frac{\sum_{k \in D_s} |C_{kk}|^{-2}}{\sum_{j = 1}^K |C_{jj}|^{-2}} \geq \frac{1}{K} \sum_{k \in D_s} |C_{kk}|^{-2}$$

Notice that the outer bound converges to the correct $\frac{E_b}{N_0 \min}$, and (37) can therefore be used as an outer bound on the slope.

Now, we want to show that $\forall \epsilon > 0, \theta$ can be made arbitrarily small. Define random variable $H_{j,\epsilon}$ as

$$H_{j,\epsilon} = \begin{cases} |C_{jj}|^{-2} & j \in D_\epsilon \\ 0 & j \notin D_\epsilon \end{cases}$$

Given the fact that $H_{j,\epsilon}$ and $H_{i,\epsilon}, i \neq j$ are independent, and $\sum_{k \in D_\epsilon} |C_{kk}|^{-2} = \sum_{j = 1}^K H_{j,\epsilon}$, we can apply the law of large number to $\theta$, which gives

$$P \left( \lim_{K \to \infty} \theta = \frac{E(H_{j,\epsilon})}{E(|C_{jj}|^{-2})} \right) = 1$$

Combining Lemma 14 and Lemma 15, we have

$$\lim_{\epsilon \to 0} E(H_{j,\epsilon}) = 0$$

This proves (27); explicitly as follows. For any $\delta > 0$ we can choose $\epsilon, \theta > 0$ sufficiently small to make (37) less than $\frac{1}{2} + \delta$. We can choose $\epsilon > 0$ sufficiently small to make $\frac{E(H_{j,\epsilon})}{E(|C_{jj}|^{-2})}$ smaller than $\theta$. Finally we can choose $K$ large enough to make $\frac{\sum_{k \in D_\epsilon} |C_{kk}|^{-2}}{\sum_{j = 1}^K |C_{jj}|^{-2}}$ smaller than $\theta$ with high probability and $\Pr(B_{\epsilon,\delta} = \emptyset)$ close to 1.

We now consider the equal power case. For $\forall \epsilon > 0$, we define

- $A_{\epsilon}$ is the event: for even $K = 2M$ users can form $M$ disjoint weak $(1 - \epsilon)$-pairs \{$(m_1, m_2), m = 1, \cdots, M$. Let the channel coefficients $C_{ij}, i, j = 1, \ldots, K$ be random variables with a distribution that could depend on $K$. We consider the following properties of this sequence of distributions

**Property 1.** $\Pr(A_{\epsilon}) \to 1$ as $K \to \infty$.

**Proposition 17.** If the channel gains $C_{ij}$ are i.i.d (independent of $K$) with continuous distribution, Property 1 is satisfied.

The proof is in the Appendix.

**Theorem 18.** If property 1 is satisfied and the direct channel gains $C_{jj}$ are i.i.d with finite 4th order moments, then in the equal power case

$$\forall \delta > 0 : \lim_{K \to \infty} \Pr \left( \Delta S_0 \leq \frac{1}{E[|C_{jj}|^4]} + \delta \right) = 1$$
Proof: For the equal power scenario where $P_j = \frac{P}{K}$, if property 1 is satisfied, then for $K = 2M$ users, $M$ disjoint weak $(1-\epsilon)$-pairs $\{m_1, m_2\}, m = 1, \cdots, M$ can be formed with high probability, and we will assume this is the case. Applying Kramer’s bound Theorem 1 on each pair, we have

$$R_{m_1} + R_{m_2} \leq \min \left( \log \left( 1 + |C_{m_1m_1}|^2 \frac{P_s}{K} \right) + \log \left( 1 + \frac{|C_{m_2m_2}|^2 P_s}{1 + |C_{m_2m_2}|^2 \frac{P_s}{K}} \right), \log \left( 1 + |C_{m_1m_2}|^2 \frac{P_s}{K} \right) + \log \left( 1 + \frac{|C_{m_1m_2}|^2 P_s}{1 + |C_{m_1m_2}|^2 \frac{P_s}{K}} \right) \right)$$

in nats/s. For each weak $(1-\epsilon)$-pair, (41) gives

$$\frac{d (R_{m_1} + R_{m_2})}{dP_s} \bigg|_{P_s=0} = \frac{|C_{m_1m_1}|^2 + |C_{m_2m_2}|^2}{K}$$

and

$$\frac{d^2 (R_{m_1} + R_{m_2})}{dP_s^2} \bigg|_{P_s=0} \geq \frac{|C_{m_1m_1}|^2 + |C_{m_2m_2}|^2 + 2 \min \left\{ |C_{m_1m_2}|^2 |C_{m_1m_1}|^2, |C_{m_2m_2}|^2 |C_{m_2m_2}|^2 \right\}}{K^2}$$

since the $M$ pairs are disjoint and using the linearity of derivatives, we have

$$\left. \frac{dR_s}{dP_s} \right|_{P_s=0} = M \sum_{m=1}^{M} \left. \frac{d (R_{m_1} + R_{m_2})}{dP_s} \right|_{P_s=0} = \sum_{j=1}^{K} |C_{jj}|^2$$

and

$$\left. \frac{d^2 R_s}{dP_s^2} \right|_{P_s=0} \geq - \sum_{m=1}^{M} \left( - \left. \frac{d^2 (R_{m_1} + R_{m_2})}{dP_s^2} \right|_{P_s=0} \right) \geq \sum_{m=1}^{M} \left( |C_{m_1m_1}|^4 + |C_{m_2m_2}|^4 + 2 (1 - \epsilon) |C_{m_1m_1}|^2 |C_{m_2m_2}|^2 \right)$$

therefore

$$E_b \frac{2}{N_0 \min} \leq \frac{K \log_e 2}{\sum_{j=1}^{K} |C_{jj}|^2}$$

$$S_s \leq 2 \sum_{m=1}^{M} \left( |C_{m_1m_1}|^4 + |C_{m_2m_2}|^4 + 2 (1 - \epsilon) |C_{m_1m_1}|^2 |C_{m_2m_2}|^2 \right)$$

$$= 2K \frac{1}{K} \sum_{j=1}^{K} |C_{jj}|^4 \left( \frac{1}{K} \sum_{j=1}^{K} |C_{jj}|^2 \right)^2$$

Now

$$\frac{1}{K} \sum_{j=1}^{K} |C_{jj}|^2 \xrightarrow{P} \mathbb{E} \left[ |C_{jj}|^2 \right]$$

$$\frac{1}{K} \sum_{j=1}^{K} |C_{jj}|^4 \xrightarrow{P} \mathbb{E} \left[ |C_{jj}|^4 \right]$$

$$\frac{1}{M} \sum_{m=1}^{M} |C_{m_1m_1}|^2 |C_{m_2m_2}|^2 \xrightarrow{P} \mathbb{E} \left[ |C_{jj}|^2 \right]$$

as $K \to \infty$ ($P$ convergence in probability) since all random variables are positive and the moments are assumed to exist. Using standard rules for convergence of transformation, we then obtain (40).
We will discuss some implications of these Theorems. For the equal rate case, (27) essentially states that the wideband slope is bounded by $\frac{1}{2}$ of that of no interference for large $K$. Since this is also achievable by Theorem 10, this is indeed the wideband slope, and delay-based interference alignment is optimum. The bound for the equal power case is slightly weaker. If the channel coefficients are IID circularly Gaussian the bound (40) for the equal power case also gives $\frac{1}{2}$; for other distributions we get a slightly weaker bound.

The Theorems have been proven under an IID assumption on all channel coefficients. This can seem restrictive and not that realistic in a line of sight model. However, the IID assumption is not essential. In Theorem 16 it is used to prove that every user has at least one other user with which it forms an $(1 - \epsilon)$-pair with high probability. This might true under many other model assumptions. It as also used to invoke the law of large numbers, which has a wide range of generalizations. In Theorem 18 the IID assumption is used to prove that users form disjoint weak $(1 - \epsilon)$-pairs, and again for invoking the law of large numbers.

What can be concluded is that, for small special examples it’s possible to find a better wideband slope by optimizing a combination of interference alignment, interference decoding, and treating interference as noise. However, it probably does not pay off to try to find a general algorithm for optimizing wideband slope, as for large $K$ it would give very little gain. Furthermore, finding such schemes are hard, because in practice it’s difficult to combine schemes. For example, in the two user case in Section III TDMA does not have soft combination with other schemes. Therefore, delay-based interference alignment is essentially optimum.

Another interesting observation is that the outer bounds do not depend on delay, only on the channel gains. Thus, the outer bound depends on the macroscopic location of nodes (e.g., if gain is proportional to $d_{ij}^{-\alpha}$ for some $\alpha > 0$), while the inner bounds depend on the microscopic location (i.e., fractional delay differences).

V. Conclusions

In this paper we have shown that by using interference alignment with delay differences, a wideband slope of half of the interference-free case is achievable. We have also shown that, mostly, it is the best achievable. What it means is that near single-user performance can be achieved in the interference channel in the low power regime. One surprising conclusion is that orthogonalizing interference is (near) optimum in the low-power regime. It is not too surprising that this is optimum in the high-SNR regime [4], since that regime is interference limited. But since the low-power regime is also noise-limited, one could have expected that orthogonalizing interference is suboptimum. That is indeed the case for a 2-user channel. But for a $K$-user channel, orthogonalizing is near optimum, as shown by Theorem 18.

A number of questions remain open. What if the bandwidth remains fixed, but the transmission rate approaches zero (e.g., in a sensor network)? This case is more complicated, and will be covered in a later paper. How can the delay based interference alignment be implemented in practical systems? As we have seen in section IV-B, the achievable spectral efficiency is very dependent on choosing the right sampling frequency, so this touches on issues of channel knowledge and estimation, and how to optimize sampling rate in a given spectral efficiency region, as well as up to what spectral efficiencies the wideband slope provides a good approximation.

REFERENCES

APPENDIX

PROOF OF LEMMA 13

Let $C_{i,e}$ be the event that user $i$ does not form an $(1 - \epsilon)$-pair with any other user. Then

$$\Pr(B_{e,\epsilon} \neq \emptyset) = \Pr\left(\bigcup_{i=1}^{K} C_{i,e}\right) \leq \sum_{i=1}^{K} \Pr(C_{i,e}) = K\Pr(C_{1,e})$$

and

$$\Pr(C_{1,e}) = \Pr\left(\forall j > 1 : \frac{|C_{j1}|^2}{|C_{11}|^2} \neq \frac{|C_{1j}|^2}{|C_{jj}|^2} \notin (1 - \epsilon, 1) \text{ and } |C_{11}|^2 \notin R_{\epsilon}\right)$$

$$= \Pr\left(\forall j > 1 : \frac{|C_{j1}|^2}{|C_{11}|^2} \notin (1 - \epsilon, 1) \text{ and } |C_{11}|^2 \notin R_{\epsilon}\right) + \Pr\left(\forall j > 1 : \frac{|C_{1j}|^2}{|C_{jj}|^2} \notin (1 - \epsilon, 1) \text{ and } |C_{11}|^2 \notin R_{\epsilon}\right) - \Pr\left(\forall j > 1 : \frac{|C_{j1}|^2}{|C_{11}|^2} \neq \frac{|C_{1j}|^2}{|C_{jj}|^2} \notin (1 - \epsilon, 1) \text{ and } |C_{11}|^2 \notin R_{\epsilon}\right)$$

$$= (1 - p_j^{K-1})\Pr\left(\forall j > 1 : \frac{|C_{j1}|^2}{|C_{11}|^2} \notin (1 - \epsilon, 1) \text{ and } |C_{11}|^2 \notin R_{\epsilon}\right) + p_j^{K-1}$$

where $p_j = \Pr\left(\frac{|C_{j1}|^2}{|C_{jj}|^2} \notin (1 - \epsilon, 1)\right) \in (0, 1)$. Notice that the events $\frac{|C_{j1}|^2}{|C_{jj}|^2} \notin (1 - \epsilon, 1)$ are independent for different $j$, and $p_j^{K-1} \rightarrow 0$. Thus, $\Pr(C_{1,e}) \rightarrow \Pr\left(\forall j > 1 : \frac{|C_{j1}|^2}{|C_{11}|^2} \notin (1 - \epsilon, 1) \text{ and } |C_{11}|^2 \notin R_{\epsilon}\right)$. As the $|C_{jj}|^2$ are independent,

$$\Pr\left(\forall j > 1 : \frac{|C_{j1}|^2}{|C_{11}|^2} \notin (1 - \epsilon, 1) \text{ and } |C_{11}|^2 \notin R_{\epsilon}\right)$$

$$= \int_{x \notin R_{\epsilon}} \prod_{j=2}^{K} \left(1 - \int_{(1 - \epsilon)x}^{x} dF_{|C_{j1}|^2}(u)\right) dF_{|C_{11}|^2}(x)$$

$$= \int_{x \notin R_{\epsilon}} \left(1 - F_{|C_{11}|^2}(x) + F_{|C_{11}|^2}((1 - \epsilon)x)\right)^{K-1} dF_{|C_{11}|^2}(x) \leq (1 - \epsilon)^{K-1}$$

$$= (1 - \mu_F(R_{\epsilon}))(1 - \epsilon)^{K-1}$$

Thus, $P_e \leq K(1 - \mu_F(R_{\epsilon}))(1 - \epsilon)^{K-1}$, and $\lim_{K \rightarrow \infty} \Pr(B_{e,\epsilon} \neq \emptyset) = 0$. 

**Proof of Lemma 14**

Given the definition \( \forall x \in \mathbb{R}_+ : F_{[C_{ii}]^2}(x) - F_{[C_{ii}]^2}((1 - \epsilon)x) \leq \hat{c} \), and given the fact \( F_{|C_{ii}|^2}(x) - F_{|C_{ii}|^2}((1 - \epsilon)x) < \epsilon_{n+1} < \epsilon_n \), it clearly follows that \( R_{\epsilon_{n+1}} \subseteq R_{\epsilon_n} \). Let \( I_{\epsilon_n}(x) \) be the indicator function of \( R_{\epsilon_n} \). Using Lebesgue dominated convergence we have

\[
\lim_{n \to \infty} \int_0^\infty I_{\epsilon_n}(x)dF_{[C_{ii}]^2}(x) = \int_0^\infty \lim_{n \to \infty} I_{\epsilon_n}(x)dF_{[C_{ii}]^2}(x) = \int_0^\infty I_0(x)dF_{[C_{ii}]^2}(x)
\]

where \( I_0(x) \) is the indicator function of the set \( R_0 = \{ x \in \mathbb{R} : F_{|C_{ii}|^2}(x) - F_{|C_{ii}|^2}((1 - \epsilon)x) = 0 \} \). Since we have assumed that \( E \left[ |C_{ii}|^{-2} \right] < \infty \), also \( \Pr(|C_{ii}|^2 = 0) = 0 \) and clearly \( \mu_\infty(R_0) = 0 \), and \( \mu(R_{\epsilon_n}) \to \mu(R_0) \).

**Proof of Lemma 15**

Given the definition of \( X_i \), we can conclude that

\[
\lim_{i \to \infty} X_i = 0 \quad \text{w.p. 1}
\]

Namely, if there is a set \( B \) with \( \mu(B) > 0 \) then \( \lim_{i \to \infty} X_i \neq 0 \) then \( B \cap \bigcap_{i=1}^\infty G_i \) which contradicts \( \lim_{i \to \infty} \mu(G_i) = 0 \).

Now \( X_i \leq X \), and therefore by Lebesgue dominated convergence

\[
\lim_{i \to \infty} E[X_i] = E[\lim_{i \to \infty} X_i] = 0
\]

**Proof of Property 1**

Model the interference channel as a graph \( G_K \), with \( K = 2M \) vertices \( u_1, u_2, \ldots, u_{2n} \). Vertices \( u_i \) and \( u_j \) are connected by edge \( E_{ij} \) if they form a weak \((1 - \epsilon)\)-pair, i.e., \((1 - \epsilon) \leq \frac{|C_{ij}|^2}{E_{ij}} \leq 1 \) or \((1 - \epsilon) \leq \frac{|C_{ij}|^2}{E_{ij}} \leq 1 \). Divide vertices into two disjoint classes \( V_1 = \{ u_1, u_2, \ldots, u_M \} \) and \( V_2 = \{ u_{M+1}, u_{M+2}, \ldots, u_{2M} \} \). Now define event

\[
\hat{A}_x = \{ \text{there exists a perfect matching in the bipartite graph } G_{M,M} \}
\]

As \( \hat{A}_x \subseteq A_x \), \( P(A_x) \geq P(\hat{A}_x) \). Thus, if we can show that \( P(\hat{A}_x) = 1 - o(1) \) as \( K \to \infty \) then Property 1 holds.

For any bipartite graph, a perfect matching exists if Hall’s condition is satisfied.

**Theorem 19** (Hall 1935). *Given a bipartite graph \( G_{M,M} \) with disjoint vertices class \( V_1 \) and \( V_2 \), \( V_1 \cup V_2 = V \), \( |V_1| = M \), whose set of edges is \( E(G_{M,M}) \), a perfect matching exists if and only if for every \( S \subseteq V_i, i = 1 \text{ or } 2 \), \( |N(S)| \geq |S| \), where \( N(S) = \{ y : xy \in E(G_{M,M}) \text{ for some } x \in S \} \).*

Any bipartite graph that does not have a perfect matching has following properties

**Lemma 20.** *Suppose \( G_{M,M} \) has no isolated vertices and it does not have a perfect matching. Then Hall’s condition must be violated by some set \( A \subseteq V_i, i = 1 \text{ or } 2 \). And such set with minimal cardinality satisfies the following necessary conditions*

\((i) \ |N(A)| = |A| - 1;
(ii) \ 2 \leq |A| \leq \left[ \frac{M}{2} \right];
(iii) \ the subgraph of \( G \) spanned by \( A \cup N(A) \) is connected, and it has at least \( 2a - 2 \) edges;
(iv) \ every vertex in \( N(A) \) is adjacent to at least two vertices in \( A \);
(v) \ any subsets of \( N(A) \) can find a perfect matching in \( |A| \).

**Proof:** (i), (ii), (iii), and (iv) are proved by Lemma 7.12 in [27], and p.82 of [28]. And (iv) is true because if there exists a subset \( B \) of \( N(A) \) that cannot find a perfect match, we could just let \( B \) be \( \hat{A} \), and its neighbours in \( A \) be \( N(\hat{A}) \). Then \( \hat{A} \) violates Hall’s condition, while \( |\hat{A}| < a \). This contradicts the assumption that \( A \) is the minimal set violating Hall’s condition.

Define the event \( F_a \): there is a set \( A \subseteq V_i, i = 1 \text{ or } 2, |A| = a \), satisfying (i), (ii) and (iii) in Lemma 20. [27] shows that for a graph with no isolated vertex, \( P(A_x) = 1 - o(1) \) is equivalent to \( P \left( \bigcup_{a=2}^{M/2} F_a \right) = o(1) \). Define \( F_1 \) as the event that there exists at least one isolated vertex in \( G_{M,M} \). In our case, we want to show that

\[
P \left( \bigcup_{a=2}^{M/2} F_a \right) + P(F_1) = o(1)
\]
Using the union bound, we have

\[
P(F_1) \leq \sum_{i=1}^{2M} P(u_i \text{ isolated}) \\
\leq 2M \cdot P(u_1 \text{ isolated}) \\
\leq (a) 2M \cdot (1 - p_{ij})^M \\
\leq (b) o(1)
\]

where \( p_{ij} \triangleq P \left( (1 - \epsilon) \leq \frac{|C_{ij}|^2}{|C_{jj}|^2} \leq 1 \right), j = M+1, \cdots, 2M \). We also define \( p_0 \triangleq P \left( (1 - \epsilon) \leq \frac{|C_{ij}|^2}{|C_{jj}|^2} \leq 1, \text{ or } (1 - \epsilon) \leq \frac{|C_{ij}|^2}{|C_{jj}|^2} \leq 1 \) for later use. (a) holds because the event \( \frac{|C_{ij}|^2}{|C_{jj}|^2} \not\in (1 - \epsilon, 1) \) is independent of \( j \). And it is a necessary condition for \( V_1 \) to be isolated.

Now, let us look into \( F_a \) for \( 2 \leq a \leq \lceil \frac{M}{2} \rceil \). Let \( A_1 \subset V_1, A_2 \subset V_2 \), and \( |A_1| = |A_2| + 1 = a \). Denote \( P(A_a) \) as the probability that the subgraph of \( G_{M,M} \) spanned by \( A_1 \cup A_2 \) satisfies (i), (ii), and (iii) in Lemma 20. We have

\[
P \left( \bigcup_{a=2}^{\lfloor \frac{M}{2} \rfloor} F_a \right) \leq \sum_{a=2}^{\lfloor \frac{M}{2} \rfloor} P(F_a) \leq 2 \sum_{a=2}^{\lfloor \frac{M}{2} \rfloor} \left( \begin{array}{c} M \\ a \end{array} \right) \left( \begin{array}{c} M \\ a-1 \end{array} \right) P(A_a) \tag{42}
\]

(d) is from the union bound; (e) is from the union bound, and from that fact that there are \( 2 \left( \begin{array}{c} M \\ a \end{array} \right) \) choices for \( A \) with \( |A| = a \), and \( \left( \begin{array}{c} M \\ a-1 \end{array} \right) \) more choices for \( N(A) \). In [27], [28], the case where edge probabilities are i.i.d, whose value is \( p \), is considered. In [27], \( P(A_a) \) is bounded using condition (i), (ii) and (iii), which gives \( P(A_a) \leq \left( \frac{a(a-1)}{2a-2} \right) p^{2a-2} p^{a(n-a+1)} \).

The term \( p^{a(n-a+1)} \) is the probability that the vertices in \( A_1 \) do not connect to vertices in \( V_2 - A_2 \). And in [28], condition (iv) instead of (iii) are used, which gives \( P(A_a) \leq \left( \frac{a}{2} \right)^{a-1} p^{2a-2} p^{a(n-a+1)} \). However, in our case, any two edges having adjacent vertices are dependent. So we use condition (v). Since for \( N(A) \), a perfect match exists, then the subgraph spanned by \( A \cup N(A) \) has \( a - 1 \) edges that are not adjacent with each other. Thus, \( P(A_a) \) can be bounded by

\[
P(A_a) \leq Pr \text{ (condition (i), (ii) and (iv) are satisfied, vertices in } A_1 \text{ do not connect to vertices in } V_2 - A_2) \tag{43}
\]

where

\[
p_{A_1,A_2} = P \left( \frac{|C_{ij}|^2}{|C_{jj}|^2} \not\in [(1 - \epsilon), 1], \text{ for all } u_i \in A_1 \text{ and } u_j \in (V_2 - A_2) \right) \tag{44}
\]

\[
\leq \prod_{u_j \in (V_2 - A_2)} P \left( \frac{|C_{ij}|^2}{|C_{jj}|^2} \not\in [(1 - \epsilon), 1], \text{ for all } u_i \in A_1 \right) \tag{45}
\]

\[
= \left( P \left( \frac{|C_{ij}|^2}{|C_{jj}|^2} \not\in [(1 - \epsilon), 1], \text{ for all } u_i \in A_1, a \times j = M_a \right) \right)^{M_a+1} \tag{46}
\]

notice that the event \( \frac{|C_{ij}|^2}{|C_{jj}|^2} \not\in [(1 - \epsilon), 1], \text{ for all } u_i \in A_1 \text{ and } u_j \in (V_2 - A_2) \) is an necessary Substitute \( |C_{jj}|^2 \) by \( x_j \), and
\[ |C_{ij}|^2 \] by \( x_1 \), denote their CDF by \( F_x(x) \), and their joint CDF \( F_X(x) \). Notice that \(|C_{ij}|^2\) are i.i.d. distributed. Then

\[
P \left( \frac{|C_{ij}|^2}{|C_{jj}|} \notin [(1 - \epsilon), 1], i = 1, \ldots, a \right) = 1 \leq \int_{A_X} dF_X(x)
\]

\[
= \int_0^\infty f_{x_{M+a}}(x_{M+a}) \prod_{i=1}^a \left( \int_{x_{M+a}}^{x_i} f_{x_i}(x_i) \, dx_i \right) \, dx_{M+a}
\]

\[
= \int_0^\infty f_{x_1}(x_1) \left( \int_{x_{M+a}}^{x_1} f_{x_{M+1}}(x_{M+1}) \, dx_{M+1} \right)^a \, dx_{M+a}
\]

\[
\leq \left( \int_0^\infty f_{x_1}^2(x_1) \, dx_1 \right)^{1/2} \left( \int_0^\infty g_{x_1}^2(x_1) \, dx_1 \right)^{1/2}
\]

where \( g(x_1) \triangleq \int_{x_{M+a}}^{x_1} f_{x_{M+1}}(x_{M+1}) \, dx_{M+1} \). (f) is from Cauchy-Schwartz inequality. Denote

\[
q_1 \triangleq \left( \int_0^\infty f_{x_1}^2(x_1) \, dx_1 \right)^{1/2} \left( \int_0^\infty g_{x_1}^2(x_1) \, dx_1 \right)^{1/2}
\]

\( q_1 < 1 \), and \( \lim_{M \to \infty} q_1 = \left( \int_0^\infty g_{x_1}^2(x_1) \, dx_1 \right)^{1/2} \). Notice that this limit value do not depend on the value of \( M \). Now combining (46) and (47), we have

\[
P(A) \leq \left( p_0 a^{-1} \prod_{k=2}^a \left( \frac{k}{1} \right) \right) q_1^{a(M-a+1)}
\]

\[
P \left( \bigcup_{a=2}^{M} F_a \right) \leq 2 \sum_{a=2}^{M} \left( \binom{M}{a} \binom{M}{a-1} \left( p_0 a^{-1} \prod_{k=2}^a \left( \frac{k}{1} \right) \right) q_1^{a(M-a+1)}
\]

\[
\leq 2 \left( \int_{\epsilon}^{\epsilon} \binom{M}{a} \binom{M}{a-1} \right)^{a-1} p_0 q_1^{a(M-a+1)}
\]

\[
\leq 2q_1 \sum_{a=2}^{M} \left( \frac{e^2 M^2}{(a-1)^2} a^2 p_0 q_1^{a-1} \right)^{a-1}
\]

\[
\leq 2q_1 \sum_{a=2}^{M} \left( 2e^2 p_0 M^2 q_1^{a} \right)^{a-1}
\]

\[
= 2q_1 \frac{M}{2} \left( 2e^2 p_0 M^2 q_1^{a} \right)^{a-1}
\]

Which means we can find a perfect matching with high probability, i.e., \( 1 - o(1) \).