Outage Capacity of the Broadcast Channel
in the Low Power Regime

Momin Uppal, Anders Høst-Madsen†, and Zixiang Xiong
Dept of Electrical and Computer Engineering, Texas A&M University, College Station, TX 77843
†Dept of Electrical Engineering, University of Hawaii, Honolulu, HI 96822
Email: momin@tamu.edu, madsen@spectra.eng.hawaii.edu, zx@ece.tamu.edu

Abstract—We consider outage capacity in the broadcast channel when the base station has none or little channel knowledge. We find the minimum energy per bit needed to achieve a certain outage probability and the corresponding wideband slope.

I. INTRODUCTION

In the papers [7], [6] we analyzed the outage capacity of the multiple access channel in the low power regime. In this paper we generalize this to the dual case of the broadcast channel. The setting is as follows. We have a base station (with one antenna) that needs to transmit $N$ independent messages to $N$ users through a fading channel. The base station has no knowledge of the fading state. For simplicity and robustness we assume that the fading is iid — essentially, the situation is that the base station know nothing about the channel. The base station has one shot: it transmit all the messages, and if they are all decoded by their respective destination, the transmission is a success, otherwise a failure. We are interested in the energy needed to stay below a certain maximum failure probability. The paper [2] analyzed energy for a fading multicast channel with ergodic capacity. We consider the broadcast channel, outage capacity, and a different channel model. Outage capacity of the broadcast channel when the base station has full channel knowledge was analyzed in [4]. A related paper for the cooperation case is [5].

II. SYSTEM MODEL

We consider an $N$-user (cooperative) Gaussian broadcast channel where all users as well as the base-station have a single transmit/receive antenna. The complex channel gain between the base-station and user $i$ is denoted by $c_i$, and two users $i$ and $j$ as $c_{ij}$ $(i,j=1,\ldots,N)$ with the assumption that $c_{ij} = c_{ji}$. Let $X[n]$ be the output of the base-station at time $n$, and $X_i[n], Y_i[n]$ be the channel input and output of user $i$, respectively. The channel can then be modeled as

$$Y_i[n] = c_i X[n] + \sum_{j=1, j\neq i}^{N} c_{ij} X_j[n] + Z_i[n]$$

where $Z_i[n]$ is IID additive white Gaussian noise with power $N_0 B$, where $B$ is the bandwidth and $N_0$ the spectral density, which without loss of generality can be assumed to be $N_0 = 1$. If users do not cooperate, $X_j[n] = 0$. We assume that if users cooperate (by transmitting information in $X_i[n] \neq 0$ to help other nodes), they do so in the half-duplex mode, i.e., they transmit and receive on different frequency bands. All channel gains $c_{ij}$ and $c_i$ are assumed to experience IID block flat fading. Furthermore, the nodes, including the base station, are assumed to have no channel state information (CSI), except as required for decoding. What a node needs to know is if it has decoded a packet correctly, so that it can forward it, which could be ensured for example by error-detection coding. A reasonable performance measure is therefore outage capacity. We assume that each user needs to receive independent information at the same rate $R_i = R$. An outage event is declared if at least one of the users cannot communicate at the target rate $R$. Let $\mathbf{R}(\mathbf{c})$ be an achievable spectral efficiency region for a specific set of channel coefficients $\mathbf{c}$. Then the outage capacity is defined as

$$R(p) = \max \{ R \mid \Pr ((R, R, \ldots, R) \notin \mathbf{R}(\mathbf{c}) \leq p \} \quad (1)$$

III. NO COOPERATION

We first consider that the nodes do not cooperate, in which case we are able to find capacity. Intuitively, one would think that in an IID fading channel the best that can be done is to transmit an equal combination (superposition, multiplexing) of messages. However, this is not the case, as will be seen shortly. First we have

Proposition 1: In the single antenna IID broadcast channel without CSI and outage capacity as performance measure, Gaussian superposition signaling is optimum.

Proof: For a specific node, an outage happens if $|c_i| < k_i$ for some $k_i$; if $|c_i| > k_i$ the node can add Gaussian noise to the output of the channel to reduce it to an equivalent channel with $|c_i| = k_i$. The point $(|c_1|, \ldots, |c_N|) = (k_1, \ldots, k_N)$ must be inside the capacity region of the (non-fading) Gaussian broadcast channel. Therefore, there is a Gaussian (superposition) code book that can be decoded at the channel state $(|c_1|, \ldots, |c_N|) = (k_1, \ldots, k_N)$, and therefore has no outage if $|c_i| > k_i$. This Gaussian code book therefore is at least as good as the original code book.

Having established that Gaussian superposition coding is optimum, we can next try to find the optimum power allocation. The proof of Proposition 1 actually gives even more insight. The optimum code book for outage capacity is a Gaussian superposition code book designed for a specific set
of coefficients $(|c_1|, \ldots, |c_N|) = (|c_1^*|, \ldots, |c_N^*|)$; to minimize outage probability, only $(|c_1^*|, \ldots, |c_N^*|)$ have to be optimized. Surprisingly, the optimum solution is asymmetric, i.e., $|c_i^*| \neq |c_j^*|$ for $i \neq j$. We illustrate this for the two-user case. In the point $(|c_1^*|, |c_2^*|)$ the rate $R$ has to satisfy

$$R \leq \min \left[ \log \left( 1 + |c_1^*|^2 \frac{\text{SNR}_1}{\text{SNR}} \right), \log \left( 1 + \frac{|c_2^*|^2 \text{SNR}_2}{1 + |c_2^*|^2 \text{SNR}} \right) \right]$$

where $\text{SNR}_j = \frac{P_j}{B}$, with $P_j$ being the power allocated to user $i$ and $B$ being the total bandwidth. We have no outage if $|c_1| \geq |c_1^*|$ and $|c_2| \geq |c_2^*|$. For Rayleigh fading, a straightforward but lengthy calculation shows that the probability of outage is minimized for

$$\text{SNR}_1 = \frac{1 - 2^{-R/2}}{2^R - 1} \text{SNR}.$$  

Therefore, there is no outage if

$$|c_1|^2 \geq |c_1^*| = \frac{2R - 1}{\text{SNR}_1} = \frac{(2R - 1)^2 2R - 1}{2^R - 1 - 2R^2/2 \text{SNR}}$$

$$|c_2|^2 \geq |c_2^*| = \frac{2R - 1}{\text{SNR} - 2R \text{SNR}_1} = \frac{(2R - 1)^2 2R - 1}{2^R - 1 - 2R \text{SNR}}$$

meaning that the optimum solution is asymmetric with each node having a different individual outage probability. On the other hand, a sub-optimum solution is to use frequency/time division multiplexing (FDM/TDM) where each user is assigned the same amount of power and frequency/time. In both cases, for a given SNR and rate $R$, an outage occurs when

$$\frac{1}{N} \log \left( 1 + \min_{i=1,\ldots,N} |c_i|^2 \text{SNR} \right) < R.$$  

(2)

For a given energy per bit, $E_b = \frac{\text{SNR}}{R^c}$, an outage in the low power regime (when $\text{SNR} \rightarrow 0$) occurs if

$$\min_{i=1,\ldots,N} |c_i|^2 < N \left( E_b \log_2 e \right)^{-1}$$

(3)

Another equivalent solution is to use symmetric superposition coding, where the code books are designed for only one threshold $c^*$ (as opposed to $N$ thresholds in the optimum asymmetric solution). In such a case, an outage occurs if $|c_i| < c^*$ for any user $i$. The superposition code book therefore satisfies

$$R = \log \left( 1 + \frac{c^* \text{SNR}_i}{1 + c^* \sum_{j=1}^{i-1} \text{SNR}_j} \right)$$

for $i = 1, \ldots, N$

with

$$\text{SNR}_1 = \frac{(1 + c^* \text{SNR})^{1/N} - 1}{c^*},$$

and

$$\text{SNR}_i = \text{SNR}_1 \left( 1 + c^* \sum_{j=1}^{i-1} \text{SNR}_j \right)$$

(4)

Notice that as opposed to the FDM/TDM/multiplexing solution, the superposition solution has to be designed for a specific $c^*$. This must be calculated based on the desired $E_b$ and outage probability. Thus, it requires exact knowledge of the fading distribution; the same is true for the optimum asymmetric solution, whereas the FDM/TDM/multiplexing solution is robust to the fading distribution. In general, the optimum asymmetric solution is difficult to find explicitly. However, in the low power regime the asymmetric solution can be characterized completely, and it is equivalent to the symmetric solution, as shown by

**Proposition 2:** For Rayleigh fading, the symmetric solution has the same $E_b|_{\text{SNR}}$ and wideband slope $[8] S_0 = \frac{2}{N}$ as the optimum solution.

**Proof:** From the proof of Proposition 1 we know that the optimum solution can be written as

$$R \leq \log \left( 1 + \frac{|c_i|^2 \text{SNR}_i}{1 + |c_i|^2 \sum_{j=1}^{i-1} \text{SNR}_j} \right)$$

for some specific $c_i^*$, and the problem is to find the optimum power allocation $\text{SNR}_i$. If we fix the spectral efficiency $R$ we then have no outage if

$$|c_i|^2 \geq \frac{2R - 1}{\text{SNR}_i - (2R - 1) \sum_{j=1}^{i-1} \text{SNR}_j}$$

Now the first part of the Proposition is obvious. An important corollary to this is that we can write

$$\text{SNR}_i = \frac{1}{N} \text{SNR} + \beta_i \text{SNR}^2 + o(\text{SNR}^2),$$

where

$$\sum_{i=1}^{N} \beta_i = 0.$$  

(5)

This is true for both the symmetric solution (4) and the optimum solution; they differ in the values of the $\beta_i$, but they both satisfy (5). We will prove that the specific values of the $\beta_i$ do not influence the wideband slope, as long as (5) is satisfied.

Notice that $2R - 1 = \alpha \text{SNR} + o(\text{SNR})$, where the $\alpha$ is the same for the symmetric and asymmetric solutions, as they have the same $E_b|_{\text{SNR}}$. The condition for no outage then is

$$|c_i|^2 \geq (2R - 1) \left( \frac{1}{N} \text{SNR} + \beta_i \text{SNR}^2 + o(\text{SNR}^2) \right)$$

$$- (\alpha \text{SNR} + o(\text{SNR})) \left( \frac{i - 1}{N} \text{SNR} \right)$$

$$+ \sum_{j=1}^{i-1} \beta_j \text{SNR}^2 + o(\text{SNR}^2) \right)^{-1}$$

$$= \frac{2R - 1}{\text{SNR} + \beta_i \text{SNR}^2 + \alpha \frac{1}{N} \text{SNR}^2 + o(\text{SNR}^2)}$$

For Rayleigh fading the outage probability is

$$P_o = \exp \left( - \sum_{i=1}^{N} |c_i|^2 \right)$$

or

$$\ln P_o = \sum_{i=1}^{N} \frac{1}{N} \text{SNR} + \beta_i \text{SNR}^2 + \alpha \frac{1}{N} \text{SNR}^2 + o(\text{SNR}^2)$$
Fixing $P_o$, the outage capacity is given as

$$R_o = \log \left( 1 - \ln(P_o) \right)$$

$$\times \left( \sum_{i=1}^{N} \frac{1}{\frac{1}{\text{SNR}} + \beta_i \text{SNR}^2 + o \frac{1}{\text{SNR}} ^2 \text{SNR}^2 + o(\text{SNR}^2)} \right)^{-1}$$

Further expanding this, it can be shown that the term in front of $\text{SNR}^2$ depends on $\beta_i$ only through $\sum_{i=1}^{N} \beta_i$, and because of (5) it is therefore independent of $\beta_i$. But that means the wideband slope is independent of the power allocation.

A. ACK Feedback

We now allow the base-station a slight amount of channel state information. We will let each node transmit one bit of feedback indicating that it has decoded its own message (let’s call it ACK) – we assume that this feedback channel is noiseless and is available for free. It turns out that the availability of this feedback yields significant performance gains in the low power regime. For comparison, we will consider the scenario where the base station has full CSI. This was considered in [4], but an explicit formula for the SNR required for a specific outage was not provided. We will first derive such a formula here. The SNR required to achieve a given spectral efficiency $R$ (in nats/s/complex dimension) can be found by solving

$$R = \ln \left( 1 + \frac{|c_i|^2 \text{SNR}}{1 + |c_i|^2 \sum_{j=1}^{N} \text{SNR}_j} \right)$$

for $i = 1, \ldots, N$ (assuming $|c_i| > |c_{i+1}|$). This gives the solution

$$\text{SNR} = \left( e^R - 1 \right) \sum_{i=1}^{N} \frac{\text{SNR}_i}{|c_i|^2} + \sum_{n=2}^{N} (e^R - 1)^n \sum_{i=1}^{N-n+1} \frac{K_{n,i}}{|c_i|^2}$$

(6)

where $K_{n,i}$ are some constants that could be explicitly found, but are not needed in the following. From (6), we have

$$\lim_{R \to 0} \frac{d\text{SNR}}{dR} = \sum_{i=1}^{N} \frac{1}{|c_i|^2}$$

$$\lim_{R \to 0} \frac{d^2\text{SNR}}{dR^2} = \sum_{i=1}^{N} \frac{1}{|c_i|^2} + 2 \sum_{i=1}^{N-1} \frac{N-i}{|c_i|^2}$$

Then using Theorem 9 in [8], the wideband slope for a specific set of channel coefficients $c$ can be calculated as

$$S_0(c) = \frac{2 \lim_{R \to 0} \frac{d\text{SNR}}{dR}}{\lim_{R \to 0} \frac{d^2\text{SNR}}{dR^2}}$$

$$= \frac{2 \sum_{i=1}^{N} \frac{1}{|c_i|^2}}{\sum_{i=1}^{N} \frac{1}{|c_i|^2} + 2 \sum_{i=1}^{N-1} \frac{N-i}{|c_i|^2}}$$

(7)

generalizing Theorem 7 in [1] to $N > 2$.

The outage wideband slope, plotted as a function of number of users in Fig. 1, can now be calculated from (7) using Theorem 2.1 in [6] (proof in [7]) reproduced here for convenience

$$S_0^{-1}(p) = \left( \int_{C(p)} \frac{1}{\|\nabla \text{SNR}_b(c)\| S_0(c) f(c) \text{d}c} \right)^{-1}$$

$$\left( \int_{C(p)} \frac{1}{\|\nabla \text{SNR}_b(c)\| f(c) \text{d}c} \right)^{-1}$$

(8)

where the integral is over the coefficients satisfying

$$\text{SNR}_b(c) = \log \left( \frac{1}{\text{SNR}} \sum_{i=1}^{N} \frac{1}{|c_i|^2} \right) = \text{SNR}_b(p).$$

The minimum energy needed to achieve a certain outage probability. In this case it can be found that

$$\|\nabla \text{SNR}_b(c)\| = \frac{10 \log e}{\ln 10 \text{SNR}_b(p)} \sum_{i=1}^{N} \frac{1}{|c_i|^8}$$

The evaluation of (8) has to be done numerically.

For ACK feedback we consider the following TDM based approach: The source dedicates all its power to transmitting the message of any one user and continues to encode and transmit using a rateless code until it receives an ACK. At that time, it allocates all its power to transmitting the message of another user (using another rateless code) and continues to do so until an ACK is received. In this fashion, the source transmits the message of each user one by one until ACKs from all users have been received, or until the maximum allowable time has elapsed, in which case an outage is declared. If $k$ is the number of bits to be transmitted per user, the time it takes to decode user $i$ is given as $k \log (1 + |c_i|^2 \text{SNR})$. Using the rateless coded strategy, the total time it will take for all users to decode their respective messages will be

$$t = \sum_{i=1}^{N} \frac{k}{\log (1 + |c_i|^2 \text{SNR})}$$

Thus the achievable rate per user is given as (a more formal argument can be given along the lines of [7])

$$C(\text{SNR}) = \frac{k}{t} = \left( \sum_{i=1}^{N} \frac{1}{\log (1 + |c_i|^2 \text{SNR})} \right)^{-1}$$

(9)

An outage obviously occurs when for a required rate $R$, $C(\text{SNR}) < R$. Translating this outage condition to the low power regime, we get the condition that an outage occurs when

$$\sum_{i=1}^{N} |c_i|^{-2} > E_b \log_2 e$$

(10)

Upon comparing (10) and (3), it is clear that a non-cooperative broadcast channel with ACK achieves a lower outage $E_b|_{\text{min}}$ than that without it. In addition, we have the following result. With ACK feedback, TDM rateless coding can achieve the same outage $E_b|_{\text{min}}$ as that of the broadcast channel with full CSI at the transmitter, and a wideband slope of $S_0 = \frac{2}{N}$. Proof: The first statement follows directly
from (10). The second statement follows by taking first and second derivatives of (9) and using Theorem 9 in [8].

Fig. 1 compares the outage wideband slope of the TDM scheme with ACK versus that with full CSI at the base station. Although the former achieves the same $E_b |_{\min}$ as the latter, it can be seen from Fig. 1, its wideband slope is much smaller. Can the wideband slope with ACK be improved beyond $S_0 = \frac{2}{3}$? An informal argument indicates that the answer is in the negative. Basically, the reason full CSI has a larger wideband slope is that superposition coding with successive decoding is used instead of TDM. However, Proposition 2 shows that without CSI, superposition coding does not have larger slope than TDM/FDM.

IV. COOPERATIVE BROADCAST CHANNEL

Cooperation is powerful method for reducing outage (cooperative diversity [3]), especially in the low power regime. In this section, we analyze how cooperation can reduce the required energy per bit in the fading broadcast channel. We will only consider achievable rate, as outer bounds on capacity are hard to find. As performance measure we consider the total energy needed by the whole system, and compare this with the no cooperation case from the previous section.

A. Cooperation without ACK Feedback

We assume that the total bandwidth is divided into $N$ orthogonal frequency bands of equal width, one for each user’s message. Thus the base-station and the users share the same frequency band to transmit a given message. Using a signal-to-noise ratio SNR, and independent rateless codes for each user, the base-station starts transmitting the messages using equal power allocation. The users continue attempting to decode their messages. If a user is successful in its decoding\(^1\), it stops listening and starts transmitting the decoded messages, except its own, with the available power distributed equally over the $N-1$ frequency bands (nothing is transmitted on the frequency corresponding to its own message) – the source meanwhile continues to transmit. We also assume that while a user is in the receive mode, it is able to detect whether something is being transmitted (or not) on a given frequency band from some other user. Thus, before it starts to transmit, the user is able to determine what other users have already decoded their corresponding messages. In addition, if $D_i$ is the set of users to have decoded their messages immediately before user $i$ does, the user, because of equal allocation of resources, is always guaranteed to decode all messages corresponding to the users in $D_i^c = \{1, \ldots, N\} / D_i$, where / indicates set subtraction. Thus, the user always transmits the messages corresponding to the users in set $D_i^c / \{i\}$.

In order to determine the outage rate, one can use the bucket-filling interpretation of [7] where each user is assigned a unit volume information bucket. A user is able to decode its messages when the information bucket is full. At any given time, let $D$ be the set of users to have decoded their messages. Then the rate of filling of the information bucket at user $i \in D^c$ is given as

$$F_i = \frac{1}{N} \log \left(1 + |c_i|^2 \text{SNR}_s + \sum_{j \in D} \frac{N|c_{ij}|^2 \text{SNR}_u}{(|D_j| - 1)(1 - R_{t_d}[j])}\right)$$

where $\text{SNR}_u$ is the average signal-to-noise ratio at the users such that $\text{SNR} = \text{SNR}_s + (N-1)\text{SNR}_u$, $t_d[j]$ is the time at which user $j$ decodes its message, and $R$ is the required rate. Note that since a user $j$ does not transmit before $t_d[j]$, the normalization factor $(1 - R_{t_d}[j])$ ensures that the users always employ all available energy when transmitting. In addition, the normalization factor $\frac{1}{|D_j| - 1}$ appears because the total available power is redistributed for transmitting $|D_j^c| - 1$ messages instead of $N$. The bucket filling approach to determining if an outage occurs or not is described below.

1: Initialize: $t=0$, $D = \emptyset$.
2: Initialize: $D_1 = \emptyset$, $U_t = \{1\}$, $t_d[i] = 0$, $i = 1, \ldots, N$.
3: While $(|D| < N$ and $t < \frac{1}{R})$
4: Update fill rates $F_i \forall i \in D^c$ using (11).
5: Calculate fill times: $t_i = U_t / F_i \forall i \in D^c$.
6: Calculate minimum fill time: $t_m = \min_{i \in D^c} t_i$.
7: Update vols: $U_t = U_t - t_m \times F_i \forall i \in D^c$.
8: Update sets: $D_j = D \setminus \{j\} \forall j \in \{i \in D^c | t_m = t_i\}$.
9: Update set: $D = D \cup \{i \in D^c | t_m = t_i\}$.
10: Update time: $t = t + t_m$.
11: Update times: $t_d[j] = t \forall j \in \{i \in D^c | t_m = t_i\}$.
12: end while
13: If $t > \frac{1}{R}$ Outage, Else Successful decoding

As SNR $\rightarrow 0$, the same algorithm can be used to determine the outage probability for a given energy per bit $E_b$ by replacing $R = 1$ and the fill rates by

$$F_i = \left(\frac{1}{N}|c_i|^2 E_b + \sum_{j \in D} \frac{|c_{ij}|^2 E_b}{(|D_j| - 1)}\right) \log_2 e,$$
where $E_{bs}$, $E_{bu}$ are the energies per bit at the source and the users, respectively, and where $E_b = E_{bs} + (N-1)E_{bu}$.

**B. Cooperation with ACK Feedback**

If a user is allowed to transmit ACK, the base-station as well as all other users are always aware if a particular user has decoded its message. Thus at any given time, the base-station as well the users in $D$ are able to redistribute their power, as well as their bandwidth to transmit only the messages of users in $D^c$. Recall that this is different than the scheme without ACK, where (a) the base-station and the users always use $N$ frequency slots (as opposed to $|D^c|$ with ACK), and (b) the base-station and the users (after they stop listening) never stop transmitting any messages. The outage rate with ACK can be evaluated using the same algorithm as the cooperation without ACK case but with a different fill rate given as

$$F_i = \frac{1}{|D^c|} \log \left( 1 + |c_i|^2 \text{SNR}_d + \frac{\sum_{j \in D} |c_i|^2 \text{SNR}_d}{1 - Rt_d[j]} \right),$$

and as $\text{SNR} \to 0$,

$$F_i = \left( \frac{1}{|D^c|} |c_i|^2 E_{bs} + \sum_{j \in D} \frac{|c_j|^2 E_{bu}}{(1 - t_d[j]) |D^c|} \right) \log_2 e, \quad (12)$$

**V. Numerical Results**

Fig. 2 shows the probability of outage versus $E_{b}\min$ for both non-cooperative and cooperative broadcast channel with and without ACK (the full CSI case has the same $E_{b}\min$ as the no cooperation scheme with ACK). It can be seen that ACK decreases the required energy considerably, both with and without cooperation. Also, for a given probability of outage, cooperation gives significant gains in the required energy per bit compared to the no cooperation case. In Fig. 3, we show the achievable outage rates versus $E_b$ for finite signal to noise ratios, which shows that similar to a cooperative MAC [7], cooperation in a broadcast channel is mainly efficient at very low rates. The curves in both Figs. 2 and 3 are obtained by numerically optimizing the power allocation amongst the base-station and the users.

**VI. Conclusion**

We have considered an IID, fading (cooperative) broadcast channel without CSI, but with the possibility of ACK feedback. Identifying outage rate as a reasonable performance measure, we analyze the performance in the low power regime by studying the minimum energy per bit and the wideband slope. There are several directions of future research, e.g. deriving outer bounds on the capacity (or minimum energy per bit and wideband slope) of a broadcast channel (a) with cooperation (b) without cooperation but with ACK, deriving explicit expressions of wideband slopes with cooperation, and last but not the least, practical coding methods.

**REFERENCES**


