Problem set 1
2.16, 2.30, 2.31, 2.35, 1.X1

Problem 1.X1
Prove Lemma 2 in section 2.9 (HINT: show that $E^\dagger E = I$).

Notice! There are some typos in the homework problems:

Problem 2.16
In the fourth line $f(A)$ should have been $f(A)$ and on the last line, $f(\lambda)$ should have been $f(A)$.

Problem 2.30
1. The identity should be:
$$\int_{-\infty}^{\infty} x^{2n} e^{-ax^2} dx = \frac{1 \cdot 3 \cdots (2n-1)}{2^n} \left( \frac{\pi}{a^{2n+1}} \right)^{1/2}$$

2. There doesn't seem to be a simple formula for the moments of an $N[\theta, \sigma^2]$ variable.
Problem set 2
Problems 3.3, 3.13 (some questions are very difficult! If you cannot find a joint sufficient statistic, find
one for each of $\theta_1, \theta_2), 3.X1

Problem 3.X1
Let $x_n$ have the PDF

$$f_p(x) = \begin{cases} \frac{1}{p_1} \exp \left( -\frac{1}{p_1} (x - p_2) \right) & x > p_2 \\ 0 & \text{otherwise} \end{cases}$$

where $p = (p_1, p_2)$. We make $M$ independent, identically distributed observations $x = (x_0, ..., x_{M-1})$, all
with distribution $f_p$.

1. Assuming $p_2$ is known, find a sufficient statistics for $p_1$.
2. Assuming $p_2$ is known, find the Minimum Variance Unbiased Estimator (MVUE) of $p_1$, and find
   the variance of the estimator.
3. Assuming $p_1$ is known, find a sufficient statistics for $p_2$.
4. Assuming $p_1$ is known, find the Minimum Variance Unbiased Estimator (MVUE) of $p_2$, and find
   the variance of the estimator.
5. Find a sufficient statistics for $p = (p_1, p_2)$, assuming neither $p_1$ nor $p_2$ known.
6. Find the MVUE of $p = (p_1, p_2)$.

You may in all cases assume that the sufficient statistic is complete.
Problem set 3 (Notice: 2 pages)

Problems 3.X1, 3.X2, 3.X3, 6.16

**Problem 3.X1 (Previous midterm question)**

The data $x[n], n = 0...N-1$ are observed. The $x[n]$'s are IID with the $\chi^2$ distribution

$$p(x[n]) = \frac{1}{\sqrt{2\pi}x[n]\sigma^2} \exp\left(-\frac{x[n]^2}{2\sigma^2}\right), \quad x[n] > 0$$

with

$$E(x[n]) = \sigma^2$$

$$\text{var}(x[n]) = 2\sigma^4$$

1. Find the CRLB for $\sigma^2$.
2. Does an efficient estimator exist? If so, what is it?
3. Find the CRLB for $\sigma$.
4. Find the Maximum Likelihood Estimator for $\sigma$.

**Problem 3.X2 - Gaussian noise**

Consider the observations

$$x_n = A + w_n, \quad n = 0...N-1$$

where $w_n$ has a Gaussian PDF with variance $\sigma^2$. The $w_n$'s are independent. The unknown parameters are $A$ and $\sigma$.

1. Find the Cramér-Rao Lower Bound for the parameters $[A \sigma]$ and $[A \sigma^2]$. Does an efficient estimator exist?
2. Find the MLE of $A$. Is the MLE unique? Is it efficient?

Now suppose that both $A$ and $\sigma$ are unknown.

3. Find the MLE for $A$ and $\sigma$ and $\sigma^2$.
4. Plot the bias of the MLE for $\sigma^2$ versus $N$.
5. Plot the CRLB and the variance of the MLE for $\sigma^2$ versus $N$.

To calculate bias and variances, you will have to do computer simulations (or Montecarlo simulations). You can do these computer simulations writing a computer program (e.g. C) or using Matlab or any other calculation program, as you prefer. Matlab has the function `randn` that generates normally distributed random numbers.

**Problem 3.X3 - Laplacian noise**

Consider the observations

$$x_n = A + w_n, \quad n = 0...N-1$$

where $w_n$ has a Laplacian PDF with variance $\sigma^2$:

$$f(w_n) = \frac{1}{\sqrt{2\sigma}} \exp\left(-\frac{\sqrt{2}|w_n|}{\sigma}\right)$$

The $w[n]$’s are independent. The unknown parameters are $A$ and $\sigma$. 

1. Find the Cramér-Rao Lower Bound for the parameters \([A \sigma]\) and \([A \sigma^2]\) and compare with the Gaussian case (Problem 7.X3). Does an efficient estimator exist?

   At first, suppose \(\sigma\) is known

2. Find the MLE of \(A\). Is the MLE unique? (Hint: Consider the median of the data).

3. Plot the bias of the MLE versus \(N\) for some choices of \(\sigma\). Compare with the estimator given by the sample mean: \(\hat{A} = \frac{1}{N} \sum_{n=0}^{N-1} x[n]\)

4. Plot the CRLB and variance of the MLE versus \(N\) for some choices of \(\sigma\). Compare with the estimator given by the sample mean: \(\hat{A} = \frac{1}{N} \sum_{n=0}^{N-1} x[n]\). Use a log-log coordinate system. How large does \(N\) have to be in order for the MLE to be efficient? Does the value of \(\sigma\) make any difference? Does the value of \(A\) make any difference?

Now suppose that both \(A\) and \(\sigma\) are unknown.

5. Find the MLE for \(A\) and \(\sigma\) and \(\sigma^2\).

6. Plot the bias of the MLE for \(\sigma^2\) versus \(N\).

7. Plot the CRLB and the variance of the MLE for \(\sigma^2\) versus \(N\).

To calculate bias and variances, you will have to do computer simulations (or Monte Carlo simulations). You can do these computer simulations writing a computer program (e.g., C) or using Matlab or any other calculation program, as you prefer. You will have to generate Laplacian distributed random numbers. Notice that if \(y\) is exponentially distributed with pdf

\[ p(y; \lambda) = \begin{cases} \lambda \exp(-\lambda y) & y > 0 \\ 0 & \text{otherwise} \end{cases} \]

and \(s\) has the distribution

\[ P(s = -1) = P(s = 1) = \frac{1}{2} \]

Then, if \(s\) and \(y\) are independent,

\[ x = s \cdot y \]

is Laplacian with variance \(2/\lambda^2\)

There exist standard methods for generating exponentially distributed random numbers, see, e.g., *numerical recipes in C* (In most calculation programs there is a function for generating exponentially distributed random numbers). Using the above you can then generate Laplacian distributed random numbers.
Problem set 4
Problems 6.10, 6.30, 6.X1

Problem 6.X1
The data $x[n], n = 0..N-1$ are observed. The $x[n]$’s are IID with the distribution

$$p(x[n]) = \begin{cases} \frac{1}{\theta} x[n]^{-\frac{1}{\theta} + 1} & x[n] \geq 1 \\ 0 & x[n] < 1 \end{cases}$$

where $\theta > 0$.

1. Find the CRLB for $\theta$.
2. Find the MLE for $\theta$. Is the MLE efficient?
3. Find a single sufficient statistic for $\theta$.
4. Assuming that the sufficient statistic is complete, find the MVU estimator for $\theta$.