# Basic Circuit Theory II
# Notes

Anders Høst-Madsen

November 9, 2009

## Useful Matlab Commands

- `help plot`: gives help on the command `plot`.
- `t=-5:0.01:5`: this generates a vector with elements from -5 to 5, with a stepsize of 0.01.
- `x=cos(t)`: this generates a vector of values of cos corresponding to the values of $t$ in the vector `t`.
- `plot(t,x)`: plot the cos-function.
- `z=3+1j*5` or `z=3+5j`: The complex number $z = 3 + 5i$.
- `abs(z),angle(z)`: modulus and phase of the complex number $z$.
- `A=[1,2;1+2j,0]`: Generates the matrix $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 1+2j & 0 \end{bmatrix}$.
- `x=A\b`: Solves the linear equation $\mathbf{Ab} = \mathbf{x}$.
- `freqs`: Plots frequency response of a circuit versus frequency. See `help freqs` for details. Notice that the frequency response has to be expressed as a rational functions of $s = j\omega$. So, for example a term $\omega^2$ this has to rewritten as $\omega^2 = -(j\omega)^2 = -s^2$.
- `residue`: partial fraction expansion.

## Partial fraction expansion with complex poles

The following addtional Laplace table entries are useful

$$\begin{aligned} f(t) &= 2|k|e^{-at}\cos(\omega t + \phi) \\ &\updownarrow \\ F(s) &= \frac{k}{s+a-j\omega} + \frac{k^*}{s+a+j\omega}, \qquad k = |k|e^{j\phi} \end{aligned}$$

and

$$\begin{aligned} f(t) &= 2|k|\frac{t^n}{n!}e^{-at}\cos(\omega t + \phi) \\ &\updownarrow \\ F(s) &= \frac{k}{(s+a-j\omega)^{n+1}} + \frac{k^*}{(s+a+j\omega)^{n+1}}, \qquad k = |k|e^{j\phi} \end{aligned}$$

# State Space

The circuit behavious is determined by the state variables $\mathbf{x}(t)$ and input $\mathbf{z}(t)$. The state variables are defined by

1. All other variables (voltages, currents) are determined by $\mathbf{x}(t)$ and $\mathbf{z}(t)$ through algebraic equations

$$y(t) = \mathbf{c}^T\mathbf{x}(t) + \mathbf{d}^T\mathbf{z}(t).$$

2. The state variables values can be choosen freely at $t = 0$, but are then uniquely determined for $t > 0$.

For circuits, usually the capacitor voltages and inductor currents can usually be choosen as state variables. However, sometimes there might exist algebraic relations between them so that 2. is not satisfied (see class example; summarized below). In that case some of the variables will need to be eliminated. In general, this can be difficult, although there exist rules for passive networks.

For a single input single output system, the transfer function is determined by

$$\begin{aligned} H(s) &= \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + D \\ (s\mathbf{I} - \mathbf{A})^{-1} &= \frac{1}{\det(s\mathbf{I} - \mathbf{A})}\operatorname{adj}(s\mathbf{I} - \mathbf{A}) \end{aligned}$$

where $\operatorname{adj}(s\mathbf{I} - \mathbf{A})$ is the *adjungate matrix* (`http://en.wikipedia.org/wiki/Adjugate_matrix`). The importat conclusion from this is that

*The denominator of $H(s)$ is "always" $\det(s\mathbf{I} - \mathbf{A})$. However, pole-zero cancellations may happen.*

It can be proven that $\det(s\mathbf{I} - \mathbf{A})$ is independent of which state variables are used. For a circuit of complexity 2, we have

$$\begin{aligned} \mathbf{A} &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\ s\mathbf{I} - \mathbf{A} &= \begin{bmatrix} s-a & -b \\ -c & s-d \end{bmatrix} \\ (s\mathbf{I} - \mathbf{A})^{-1} &= \frac{1}{\det(s\mathbf{I} - \mathbf{A})}\begin{bmatrix} s-d & b \\ c & s-a \end{bmatrix} \end{aligned}$$

## Example

Consider the circuit in Figure 1. The obvious guess for state variables are the capacitor voltages and inductor current $v_1, v_2, i_L$. However

$$v_1(t) + v_2(t) = v_s(t)$$

and they therefore contradict condition 2. We therefore eliminate one of $v_1, v_2$. As state variables we then choose $v_1, i_L$. KVL gives

$$v_s(t) = v_1(t) + L\frac{di_L(t)}{dt} + Ri_L(t)$$

and KCL gives

$$\begin{aligned} i_L(t) &= C_1\frac{dv_1(t)}{dt} - C_2\frac{dv_2(t)}{dt} \\ &= C_1\frac{dv_1(t)}{dt} - C_2\frac{d(v_s(t) - v_1(t))}{dt} \end{aligned}$$

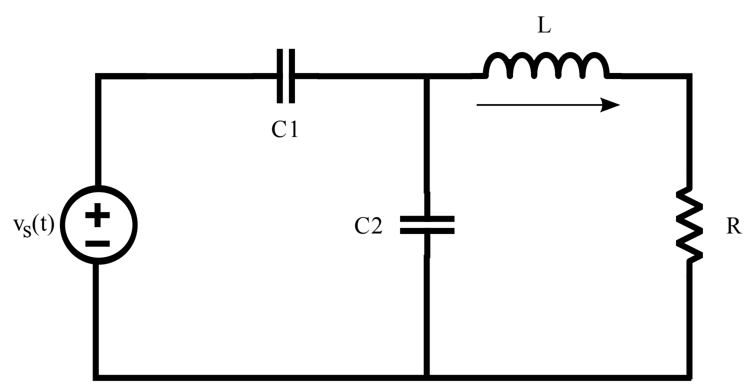


Figure 1:

We then get the state space equations

$$\begin{bmatrix} \frac{di_L(t)}{dt} \\ \frac{dv_1(t)}{dt} \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C_1+C_2} & 0 \end{bmatrix} \begin{bmatrix} i_L(t) \\ v_1(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{L} & 0 \\ 0 & \frac{C_2}{C_1+C_2} \end{bmatrix} \begin{bmatrix} v_s(t) \\ \frac{dv_s(t)}{dt} \end{bmatrix}$$

## Stability

The textbook is incorrect. A signal $x(t)$ is said to be bounded if $|x(t)| < K$ for all $t$ and some constant $K$. A *system* is stable if

> *Any bounded input results in a bounded output, i.e., if the input satisfies $|x(t)| < K_x$ for some $K_x$ the output $y(t)$ satifies $|y(t)| < K_y$ for some constant $K_y$ ($K_y$ may depend on $K_x$.)*

It can be proven that a system is stable if and only if the impulse response $h(t)$ satisfies

$$\int_0^\infty |h(t)|dt \quad < \quad \infty$$

In terms of $H(s)$ a system is stable if and only if

> *All the poles of $H(s)$ is in the left half plane, i.e., every pole $p_i$ satisfies $\Re(p_i) < 0$. Additionally in $H(s) = \frac{N(s)}{D(s)}$ the degree of $N(s)$ is less than or equal to the degree of $D(s)$.*

Consider now a system described by the state equations

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{z}(t)\\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{z}(t)\end{aligned}$$

The system is said to be stable if every bounded input $\mathbf{z}(t)$ results in bounded state variables, that is, if $|\mathbf{z}(t)| < K_z$ then $|\mathbf{x}(t)| < K_x$. This can be shown to be true if and only if all roots of $\det(s\mathbf{I} - \mathbf{A})$ are in the left half plane. This with knowledge of linear algebra will knwo that this is equivalent to requiring all eigenvalues of $\mathbf{A}$ to have negative real part.

Let us compare the two definitions. Since the denominator of $H(s) = \frac{N(s)}{D(s)}$ is $\det(s\mathbf{I} - \mathbf{A})$ the two criteria seem are almost equivalent. However, since pole-zero cancellations can happen in $H(s) = \frac{N(s)}{D(s)}$, is not always possible to see from a specific output if a the whole system is stable. There is one further issue. When deriving the state equations, some of the original inputs to the system may turn into derivatives or integrals in $\mathbf{z}(t)$ (see class example). Therefore, although the system may be stable in the state space formulation, it may not be stable with respect to the *original* inputs. To say that a circuit is stable, we therefore require

1. All roots of$\det(s\mathbf{I} - \mathbf{A})$ are in the left half plane, or equivalently, all eigenvalues of $\mathbf{A}$ have negative real part.

2. The input $\mathbf{z}(t)$ to the state model are the original inputs.