There are 8 question with a total score of 70p. For all questions, you need show your work, e.g., how you solve linear equations, find partial fraction expansions etc. Just stating the results does not give full credit.

**Problem 1**

Find the Laplace transform of the following function

1. (5p) \( x_2(t) = e^t(u(t) - u(t - 2)). \)

**Solution**

1. We can write \( x_2(t) = e^t u(t) - e^{t-2} u(t-2). \) We then use linearity and timeshift

\[
X_2(s) = \frac{1}{s-1} + e^{2}e^{-2s} \frac{1}{s-1}
\]

**Problem 2**

1. (10p) Find the inverse Laplace transform \( f_1(t) \) corresponding to \( F_1(s) = \frac{1}{s^2 + s - 12} \)
2. (10p) Calculate \( f_2(t) = f_1(t) \ast (e^{-(t-1)} u(t-1)), \) where \( \ast \) denotes convolution.

**Solution**

1. The poles are

\[
p = \frac{-1 \pm \sqrt{1 - 4(-12)}}{2} = \{3, -4\}
\]

So

\[
F_1(s) = \frac{A}{s-3} + \frac{B}{s+4}
\]
with

\[
A = \left. \frac{1}{s+4} \right|_{s=3} = \frac{1}{7}
\]

\[
B = \left. \frac{1}{s-3} \right|_{s=-4} = -\frac{1}{7}
\]

so

\[
F_1(s) = \frac{1}{7} \frac{1}{s-3} - \frac{1}{7} \frac{1}{s+4}
\]

and

\[
f_1(t) = \frac{1}{7} e^{3t} - \frac{1}{7} e^{-4t}
\]

2. In the Laplace domain

\[
F_3(s) = \frac{1}{(s-3)(s+4)} \frac{1}{s+1} e^{-s}
\]

We have

\[
\bar{F}_3(s) = \frac{1}{(s+1)(s-3)(s+4)} = \frac{A}{s+1} + \frac{B}{s-3} + \frac{C}{s+4}
\]

with

\[
A = \left. \frac{1}{(s-3)(s+4)} \right|_{s=-1} = -\frac{1}{12}
\]

\[
B = \left. \frac{1}{(s+1)(s+4)} \right|_{s=3} = \frac{1}{28}
\]

\[
C = \left. \frac{1}{(s+1)(s-3)} \right|_{s=-4} = \frac{1}{21}
\]

Therefore

\[
f_3(t) = -\frac{1}{12} e^{-(t-1)} u(t-1) + \frac{1}{21} e^{-4(t-1)} u(t-1) + \frac{1}{28} e^{3(t-1)} u(t-1)
\]

### Problem 3

A circuit is characterized by the following differential equation

\[
\frac{d^2 y(t)}{dt^2} - 10 \frac{dy(t)}{dt} + 25 y(t) = \frac{d^3 x(t)}{dt^3}
\]

1. (5p) Find the transfer function of the circuit.

2. (15p) If the input to the circuit is \(x(t) = u(t)\) and the initial conditions are \(y(0-) = 0, y'(0-) = 0\), what is the output of the circuit?
Solution

1. The transfer function is

\[ H(s) = \frac{s^3}{s^2 - 10s + 25} \]

2. The output is

\[ Y(s) = \frac{s^3}{s^2 - 10s + 25} \cdot \frac{1}{s^2} = \frac{s^2}{s^2 - 10s + 25} \]

First

\[ Y(s) = \frac{s^2}{s^2 - 10s + 25} = 1 + \frac{As + B}{s^2 - 10s + 25} \]

or

\[ \frac{s^2}{s^2 - 10s + 25} = \frac{s^2 - 10s + 25 + As + B}{s^2 - 10s + 25} \]

so

\[ Y(s) = 1 + \frac{10s - 25}{s^2 - 10s + 25} \]

The poles are

\[ p = \frac{10 \pm \sqrt{10^2 - 4 \times 25}}{2} = 5 \]

so

\[ \tilde{Y}(s) = \frac{A}{s - 5} + \frac{B}{(s - 5)^2} \]

where

\[ B = \left. 10s - 25 \right|_{s=5} = 25 \]
\[ A = \left. \frac{d}{ds} (10s - 25) \right|_{s=5} = 10 \]

So

\[ y(t) = \delta(t) + 10e^{5t} + 25te^{5t} \]

Problem 4

Consider the following circuit
The input voltage is \( v_i(t) = u(t) \). The initial voltage over the capacity is 0, while the initial current through the inductor is \( i_L(0^-) = -1 \text{A} \).

1. (10p) Redraw the circuit in the Laplace domain.

2. (10p) Find the output voltage \( V_o(s) \) in the Laplace domain.

3. (5p) Find the output voltage \( v_o(t) \) in the time domain.

Solution

1. Place a current source in parallel with the inductor.

2. Use superposition. The voltage due to the input is

\[
V_{o,1}(s) = \frac{0.5 \times 9s}{0.5 + 9s} \frac{1}{s} = \frac{4.5s \times 5s}{4.5s \times 5s + 0.5 + 9s} \frac{1}{s} = \frac{22.5s}{22.5s^2 + 9s + 0.5} = \frac{45s}{45s^2 + 18s + 1}
\]

The voltage due to the initial conditions is

\[
V_{o,2}(s) = \frac{1}{2 + \frac{1}{15} + 5s} \frac{1}{s} = \frac{9}{45s^2 + 18s + 1}
\]

So, the total voltage is

\[
V_o(s) = \frac{45s + 9}{45s^2 + 18s + 1}
\]

3. We first write

\[
V_o(s) = \frac{s + \frac{1}{3}}{s^2 + \frac{2}{4} s + \frac{1}{45}} = \frac{s + \frac{1}{3}}{s + \frac{1}{3} \left( s + \frac{1}{15} \right)} = \frac{A}{s + \frac{1}{3}} + \frac{B}{s + \frac{1}{15}}
\]
where

\[
A = \left. \frac{s + \frac{1}{5}}{s + \frac{1}{15}} \right|_{s = -\frac{4}{5}} = \frac{1}{2}
\]

\[
B = \left. \frac{s + \frac{1}{5}}{s + \frac{1}{3}} \right|_{s = -\frac{4}{5}} = \frac{1}{2}
\]

Therefore

\[
v_o(t) = \frac{1}{2} e^{-\frac{4}{5}t} + \frac{1}{2} e^{-\frac{4}{15}t}
\]