EE640 Fall 2007

Problem Set 4 Solutions

0) Find the MMSE estimate of X and the resulting MSE given X and Y are jointly Gaussian. Find posterior density given by

$$f_{X|Y}(x|y) = f_{X,Y}(x,y)/f_Y(y)$$

$$= \frac{1/(2\pi\sqrt{\sigma_X^2\sigma_Y^2(1-\rho^2)})\exp(-1/(2(1-\rho^2))((x-m_X)^2/\sigma_X^2+(y-m_Y)^2/\sigma_Y^2-2\rho(x-m_X)(y-m_Y)/(\sigma_X\sigma_Y)))}{1/(\sqrt{2\pi\sigma_Y^2})\exp(-(y-m_Y)^2/(2\sigma_Y^2))}$$

$$= 1/(\sqrt{2\pi\sigma_X^2(1-\rho^2)})\exp(-(x-m_X-\sigma_{XY}(y-m_Y)/\sigma_Y{}^2)^2/(2\sigma_X^2(1-\rho^2)))$$

where $\rho = \sigma_{XY}/(\sigma_X\sigma_Y)$. Note that the posterior density is Gaussian and therefore $\hat{X}(Y) = \mathbf{E}(X|Y) = m_X + \sigma_{XY}(y-m_Y)/\sigma_Y{}^2$ and the MSE is the expectation of the conditional variance given by $\sigma_X^2\sigma_Y^2(1-\rho^2)$.

1)

a) $\hat{X}(Y) = \sigma_{XY}/\mathbf{VAR}(Y)Y = 2/3Y$ as $\sigma_{XY} = 2$ and $\mathbf{VAR}(Y) = 3$. The MSE is given by

$$\mathbf{E}[(X-\hat{X}(Y))X] = 2 - \mathbf{E}(2/3XY) = 2/3$$

b) Need to find $\mathbf{E}(X|Y)$. First find $f_{X|Y}(x|y) = f_{Y|X}(y|x)f_X(x)/f_Y(y)$. Note that $f_{Y|X}(y|x)$ is a Gaussian density $Y = X + N$ and the pdf of X is a convolution of a Gaussian curve with two unit impulses at 1 and -1. The pdf of Y is also a convolution of a Gaussian curve with two unit impulses at 1 and -1. We then have that

$$f_{X|Y}(x|y) = \frac{\left(1/(2\sqrt{2\pi})\exp(-(x-1)^2/2)+1/(2\sqrt{2\pi})\exp(-(x-1)^2/2)\right)1/(\sqrt{2\pi})\exp(-(y-x)^2/2)}{1/(2\sqrt{4\pi})\exp(-(y-1)^2/4)+1/(2\sqrt{4\pi})\exp(-(y+1)^2/4)}$$

We can then find the conditional expectation to get that

$$\mathbf{E}(X|Y) = \frac{((y+1)/2)\exp(-(y-1)^2/4)+((y-1)/2)\exp(-(y+1)^2/4)}{\exp(-(y-1)^2/4)+\exp(-(y+1)^2/4)} = y + \arctan(y/2)$$

2)

a) Note that X and Y have the same marginal with $f_X(x) = f_Y(x) = (.5+x)(u(x)-u(x-1)$. The conditional density is given by

$$f_{X|Y}(x|y) = \frac{x+y}{1/2+y} \quad 0 < x < 1, 0 < y < 1$$

Then $\hat{X}_{MMSE}(Y) = E(X|Y) = \int_0^1 x(x+y)/(1/2+y)dx = (3y+2)/(6y+3)$. Similarly we can calculate $\mathbf{VAR}(X|Y) = (3+4y)/(6+12y) - (3y+2)^2/(6y+3)^2$. The MSE is then found by averaging the $\mathbf{VAR}(X|Y)$ to get

$$MSE = \int_0^1 \frac{6y^2+6y+1}{18(2y+1)^2}(1/2+y)dy = 1/12 - \log(3)/144$$

If we compare to linear estimate we have $\mathbf{E}(X) = \mathbf{E}(Y) = 7/12$, $\mathbf{VAR}(X) = \mathbf{VAR}(Y) = 11/144$, $\rho = -1/11$ and $\hat{X}_{LMMSE}(Y) = -1/11(Y - 7/12) + 7/12 = -Y/11 + 7/11$. The $LMSE = 5/66$. We then have $MSE/LMSE = .99929$ indicating that although the estimates are different there is very little difference between the MMSE estimate and the LMMSE estimate.

b) Here $\hat{X}_{MMSE}(Y) = \mathbf{E}(X|Y) = |Y|$ and we have $MSE = 0$. Note $\mathbf{E}(X) = \sqrt{2/\pi}$, $\mathbf{E}(X^2) = 1$, $\mathbf{E}(Y) = 0$, $\mathbf{E}(Y^2) = 1$, $\rho = 0$. We therefore have $\hat{X}_LMMSE(Y) = \sqrt{2/\pi}$ and $LMSE = 1 - 2/\pi$. Here we have $MSE/LMSE = 0$.

3)

a) We achieve minimization of $\mathbf{E}(|X - \hat{X}(Y)|)$ when we minimize inner expectation given by $C(\hat{X}(Y)) = \mathbf{E}(|X - \hat{X}(Y)||Y)$. Taking derivative we get that

$$dC(\hat{X}(Y))/d\hat{X}(Y) = d/d\hat{X}(Y)(\int_{-\infty}^{\infty} |x-\hat{X}(y)| f_{X|Y}(x|y)dx) = \int_{-\infty}^{\hat{X}(Y)} f_{X|Y}(x|y)dx - \int_{\hat{X}(Y)}^{\infty} f_{X|Y}(x|y)dx$$

Setting the derivative equal to zero and solving for $\hat{X}(Y)$ we get that $\hat{X}_{ABS}(Y) =$ median of conditional pdf $f_{X|Y}(x|y)$ or where $F_{X|Y}(x|y) = .5$.

b) Note that

$$P(|X - \hat{X}(Y)| > \epsilon) = \mathbf{E}(I(|X - \hat{X}(Y)| > \epsilon) = 1 - \int_{-\infty}^{\infty} \int_{\hat{X}(Y)-\epsilon}^{\hat{X}(Y)+\epsilon} f_{X|Y}(x|y)dx f_Y(y)dy$$

To minimize expression when ϵ is small maximize right hand side inner integral. This gives $\hat{X}_{MAP}(Y) = \arg\max_x f_{X|Y}(x|y)$ or the mode of the conditional pdf.

c) The posterior density is a gamma (2) RV with

$$f_{X|Y}(x|y) = x(y+1)^2 e^{-x(y+1)} u(x)u(y)$$

as the marginal of Y is given by $f_Y(y) = u(y)/(y+1)^2$. Then $\hat{X}_{MMSE}(Y) = 2/(Y+1)$, $\mathbf{VAR}(X|Y) = 2/(y+1)^2$ and MSE $= 2/3$.

The MAP estimate is easy to get as we maximize $f_{X|Y}(x|y)$ or equivalently the $\log(f_{X|Y}(x|y))$ to get that $\hat{X}_{MAP}(Y) = 1/(Y+1)$.

The ABS estimate can be found by evaluating the conditional posterior distribution, $F_{X|Y}(x|y) = .5$ and solving for x. We have that

$$F_{X|Y}(x|y) = 1 - (y+1)x \exp(-x(y+1)) + \exp(-x(y+1))$$

and we then get that $\hat{X}_{ABS}(Y) \approx 1.257/(Y+1)$. Note all three estimates are different.

4) Key to solving problem is expanding $Y(k) = \sum_0^\infty a_{k+i} 2^{-i-1}$ and a_k are iid Bernoulli RV with parameter $p = .5$. Note that $\mathbf{E}(a_k) = .5$ and $\mathbf{VAR}(a_k) = .25$.

a) Using the expression for $Y(k)$ above we get that $\mathbf{E}(Y(k)) = .5$ and assume that $i \geq j$ then

$$COV(Y(i)Y(j)) = .25 \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} COV(a_{k+i} a_{l+j}) 2^{-(k+l)}$$

Note that the a_k are all independent so that $COV(a_{k+i}a_{l+j}) = .25\delta(l+j-k-i)$ and therefore

$$COV(Y(i)Y(j)) = 1/16\sum_{l=0}^{\infty} 2^{-2l}2^{-(i-j)} = 2^{-(i-j)}/12$$

In general we get that $COV(Y(i)Y(j)) = 2^{-|i-j|}/12$.

b) Note that $Y(k)$ is Markov so $\hat{Y}_k$ only depends on most recent observation $Y(k-1)$ as other observations are not relevant. We then get that $\hat{Y}(k) = (1/24)/(1/12)(Y(k-1) - .5) + .5 = Y(k-1)/2 + 1/4$.

c) Note that $\rho = (1/24)/(1/12) = .5$ and therefore $MSE = \mathbf{VAR}(Y(k))(1-\rho^2) = 1/16$.

d) Yes, best predictor has zero MSE and is given by $\hat{Y}(k) = 2Y(k-1)\text{mod}(1)$.

5)

a) $Z = AY$ which implies $m_Z = Am_Y$ and

$$\Lambda_Z = \mathbf{E}[(Z-m_Z)(Z-m_Z)^T] = \mathbf{E}[A(Y-m_Y)(Y-m_Y)^T A^T] = A\Lambda_Y A^T$$

We then get ${\Lambda_Z}^{-1} = A^{-1T}{\Lambda_Y}^{-1}A^{-1}$. It then follows that

$$\hat{X}_{LMMSE}(Z) = [\mathbf{E}[(Z-m_Z)X]]^T {\Lambda_Z}^{-1}(Z-m_Z) + m_X$$

$$= [E(AYX) - Am_Y m_X]^T A^{-1}{\Lambda_Y}^{-1}A^{-1}(AY - Am_Y) + m_X$$

$$= [\mathbf{E}[(Y-m_Y)X]]^T {\Lambda_Y}^{-1}(Y-m_Y) + m_X = \hat{X}_{LMMSE}(Y)$$

b) Let $m_X = 0$. since Y and Z are uncorrelated $\sigma_{YZ} = 0$. Then we have that

$$\hat{X}_{LMMSE}(Y,Z) = (\mathbf{E}(XY), \mathbf{E}(XZ))\begin{bmatrix} \sigma_Y^2 & 0 \\ 0 & \sigma_Z^2 \end{bmatrix}^{-1}\begin{bmatrix} Y - m_Y \\ Z - m_Z \end{bmatrix}$$

$$= \mathbf{E}(XY)(\sigma_Y^2)^{-1}(Y-m_Y) + \mathbf{E}(XZ)(\sigma_Z^2)^{-1}(Z-m_Z) = \hat{X}(Y)_{LMMSE} + \hat{X}(Z)_{LMMSE}$$

6)

a) Use projection theorem to get $\mathbf{E}(\epsilon_n) = \mathbf{E}(Y_n - \hat{Y}_{n+1|n}) = 0$ as error is orthogonal to a constant. Let $n > k$, then $\mathbf{E}(\epsilon_n\epsilon_k) = 0$ as ϵ_k is a linear combination of $Y_0 \dots Y_k$ and we can again apply projection theorem.

b-c) From a) we have that ϵ_n is zero mean and uncorrelated. We then have that $\Lambda_{\mathcal{E}_n}$ is a diagonal matrix with the variances of ϵ_n on the diagonals. The inverse matrix is also diagaonal with the inverse of the variances of ϵ_n on the diagonals.

We then use 5a) to get an estimate of $\hat{Y}_{n+1|n}$. Note that $\mathcal{E}_n$ is related to $\mathbf{Y}_n$ via a linear transformation and therefore can find $\hat{Y}_{n+1|n}$ by using the innovations sequence $\mathcal{E}_n$. First note that $\hat{Y}_{n+1|-1} = \mathbf{E}(Y_{n+1})$ as this is an estimate given no data. We also have that

$$\hat{Y}_{n+1|n} = [\mathbf{E}(Y_{n+1}\epsilon_0), \dots, \mathbf{E}(Y_{n+1}\epsilon_n)](\Lambda_{\mathcal{E}_n})^{-1}\mathcal{E}_n + \hat{Y}_{n+1|-1} = \sum_{i=0}^{n}\mathbf{E}(Y_{n+1}\epsilon_i)(\mathbf{E}({\epsilon_i}^2))^{-1}\epsilon_i + \hat{Y}_{n+1|-1}$$

$$= \hat{Y}_{n+1|n-1} + \mathbf{E}(Y_{n+1}\epsilon_n)\mathbf{E}((\epsilon_n)^2)^{-1}\epsilon_n. \quad n = 0, 1, \ldots$$

Note that $\epsilon_n = Y_n + \hat{Y}_{n|n-1}$ and therefore using the above equation for $\hat{Y}_{n+1|n}$ we get that $\mathbf{Y}_n = A\mathcal{E}_n + \mathbf{E}(\mathbf{Y}_n)$ where A is a lower triangular matrix with ones on the diagonals and

$$A_{i,j} = \mathbf{E}(Y_{i-1}\epsilon_{j-1})[\mathbf{E}({\epsilon_{j-1}}^2)]^{-1}\epsilon_{j-1}, \;\; 1 \leq j < i \leq n+1$$