#6-20
(a) yes: \( f_x(x) = xe^{-x}, f_y(y) = e^{-y}, 0 < x < \infty, 0 < y < \infty \)

(b) no: \( f_x(x) = \int_x^1 f(x,y)dy = 2(1-x), 0 < x < 1 \)
\[
f_y(y) = \int_y^0 f(x,y)dx = 2y, 0 < y < 1
\]

#6-23
(a) yes: \( f_x(x) = 12x(1-x) \int_0^1 ydy = 6x(1-x), 0 < x < 1 \)
\[
f_y(y) = 12y \int_0^1 x(1-x)dx = 2y, 0 < y < 1
\]

(b) \( E[X] = \int_0^1 6x^2(1-x)dx = 1/2 \)

(c) \( E[Y] = \int_0^1 2y^2 dy = 2/3 \)

(d) \( Var(X) = \int_0^1 6x^3(1-x)dx - 1/4 = 1/20 \)

(e) \( Var(X) = \int_0^1 2y^3 dy - 4/9 = 1/18 \)

#6-39
(a) \( P\{X = j, Y = i\} = \frac{1}{5}, j = 1, \ldots, 5, i = 1, \ldots, j \)

(b) \( P\{X = j | Y = i\} = \frac{1/(5j)}{\sum_{k=i}^{5} 1/(5k)} = \frac{1/j}{\sum_{k=i}^{5} 1/k}, 5 \geq j \geq i \)

(c) No.

#6-42
(a) \( f_{X|Y}(x|y) = \frac{xe^{-x(y+1)}}{\int_0^\infty xe^{-x(y+1)}dx} = (y+1)^2 xe^{-x(y+1)}, 0 < x \)

(b) \( f_{Y|X}(y|x) = \frac{xe^{-x(y+1)}}{\int_0^y xe^{-x(y+1)}dy} = xe^{-xy}, 0 < y \)
\[ P\{XY < a\} = \int_0^{\frac{a}{y+1}} \int_0^\infty x e^{-x(y+1)} \, dy \, dx = \int_0^\infty (1-e^{-a}) e^{-x} \, dx = 1-e^{-a} \]

\[ f_{XY}(a) = e^{-a}, 0 < a \]
2) This problem is similar to the practice exam problem.

a) We have $P(X = k) = \prod_{i=1}^{k-1} (3-i+1)(6)/((9-i+1)(9-k+1)$ for $k = 1, \ldots, 4$. Therefore $P(X = 1) = 2/3$, $P(X = 2) = 1/4$, $P(X = 3) = 1/14$, $P(Y = 4) = 1/84$. To compute the joint pmf of $X$ and $Y$ we need to compute conditional pmf given by

$$P(Y = j|X = k) = \prod_{i=1}^{j-1} (4-i+k)/(8-i+1)(8-j+1), \quad 1 \leq j \leq 5-k, \quad k = 1, 2, 3, 4$$

The joint pmf is given by $p_{X,Y}(j,k) = P(Y = j|X = k)P(X = k)$ where

$$p_{X,Y}(1,1) = 5/12, \quad p_{X,Y}(1,2) = p_{X,Y}(2,1) = 5/28$$

$$p_{X,Y}(3,1) = p_{X,Y}(2,2) = p_{X,Y}(1,3) = 5/84$$

$$p_{X,Y}(4,1) = p_{X,Y}(3,2) = p_{X,Y}(2,3) = p_{X,Y}(1,4) = 1/84$$

The marginal pmf of $Y$ is the same as $X$.

b) $X$ and $Y$ are not independent.

c) We get that $m_X = m_Y = 10/7$, \(\text{VAR}(X) = \text{VAR}(Y) = 45/98\), \(\text{E}(XY) = 55/28\), and \(\text{COV}(X, Y) = -15/196\).

d) To simulate on matlab, first generate a permutation of the numbers 1 through 9. Then we can simulate drawing balls from an urn without replacement. Matlab code is below.

```matlab
n=10000; k=1:9; times=zeros(6,n);
u9= rand(9,n);
[a,perm] = sort(u9);
sample = [ones(1,6) zeros(1,3)];
blue = sample(perm);
for i=1:n,
times(:,i) = k(blue(:,i)==1);
end
sojourn=diff([zeros(1,n);times]);
x=sojourn(1,:); y=sojourn(2,:);
mx=mean(x);my=mean(y); sx=var(x);sy=var(y);
cov=mean(x.*y) -mx*my; rho=cov/(sqrt(sx*sy));
```

The averages computed on matlab are 1.4277 for mean of $X$, 1.4298 for mean of $Y$, .4586 for variance of $X$, .4599 for variance of $Y$, and -.0718 for covariance of $X$ and $Y$. The correlation coefficient is -.1564.
3)

a) This was done in class. There are two methods. You can either either use moment generating functions or convolve pdfs. We have that \( M_Z(t) = \frac{\lambda^2}{\lambda - t} \). This is MGF of a Gamma RV with parameter \( k = 2 \) with pdf \( p_Z(z) = \lambda^2z \exp(-\lambda z)u(z) \).

b) Again can use MGF or work with pdfs. Note that

\[
M_W(t) = \mathbb{E}(\exp((X - Y)t)) = M_X(t)M_Y(-t) = \frac{\lambda^2}{(\lambda^2 - t^2)} = .5\lambda/(\lambda - t) + .5\lambda/(\lambda + t)
\]

We can take inverse transform to get that \( p_W(w) = .5\lambda \exp(-\lambda |w|) \).

c) Here we work with CDF. We have that

\[
F_V(v) = P(V \leq v) = P(\max(X, Y) \leq v) = P(X \leq v, Y \leq v) = (1 - \exp(-\lambda v))u(v).
\]

To find pdf differentiate CDF to get that \( p_V(v) = 2\lambda \exp(-\lambda v)(1 - \exp(-\lambda v))u(v) \).

To simulate using matlab generate two exponential RVs and perform function. lambda = 1; n=100000;
x = -1/lambda *log(rand(1,n)); y = -1/lambda *log (rand(1,n));
z=x+y, w=x-y; v = max(x,y);

![Sample PDF of Z](image1.png)

![Sample CDF of Z](image2.png)

**Figure 1: pdf of Z**
Figure 2: pdf of $W$

Figure 3: pdf of $V$