EE342 Spring 2007
PS 5 solutions

#5-1
(a) \( e^{\int_{1}^{c} (1-x^2) \, dx} = 1 \Rightarrow c = 3/4 \)

(b) \( F(x) = \frac{3}{4} \int_{1}^{c} (1-t^2) \, dt = \frac{3}{4} \left( \frac{x^3}{3} + \frac{2}{3} \right), \quad -1 < x < 1 \)

#5-6
(a) \( E[X] = \frac{1}{4} \int_{0}^{\infty} x^2 e^{-x/2} \, dx = 2 \int_{0}^{\infty} y^2 e^{-y} \, dy = 2 \Gamma(3) = 4 \)

(b) By symmetry of \( f(x) \) about \( x = 0 \), \( E[X] = 0 \).

(c) \( E[X] = \int_{5}^{\infty} \frac{5}{x} \, dx = \infty \).

#5-11
\( X \) is uniform on \((0, L)\).

\[
P\left\{ \min\left( \frac{X}{L-X}, \frac{L-X}{X} \right) < \frac{1}{4} \right\} = 1 - P\left\{ \min\left( \frac{X}{L-X}, \frac{L-X}{X} \right) > \frac{1}{4} \right\} = 1 - P\left\{ \frac{X}{L-X} > \frac{1}{4}, \frac{L-X}{X} > \frac{1}{4} \right\} = 1 - P\left\{ \frac{X}{L-X} > \frac{4L}{5}, X < \frac{4L}{5} \right\} = 1 - P\left\{ \frac{L}{5} < X < \frac{2L}{5} \right\} = 1 - \frac{3}{5} = \frac{2}{5} \)

#5-15
(a) \( P\{X > 5\} = 1 - \Phi\left( \frac{5-10}{6} \right) = \Phi(0.8333) = .7977 \)

(b) \( P\{4 < X < 16\} = \Phi\left( \frac{16-10}{6} \right) - \Phi\left( \frac{4-10}{6} \right) = 2\Phi(1) - 1 = .6827 \)

(c) \( P\{X < 8\} = \Phi(0.3333) = .3695 \)

(d) \( P\{X < 20\} = \Phi(1.6667) = .9522 \)

(e) \( P\{X > 16\} = 1 - \Phi(1) = .1587 \)

#5-20
Let \( X \) denote the number in favor. Then \( X \) is binomial with mean 65 and standard deviation \( \sqrt{65(.35)} \approx 4.77 \). Also let \( Z \) be a standard normal random variable.

(a) \( P\{X \geq 50\} = P\{X \geq 49.5\} = P\left[ \frac{X - 65}{4.77} \geq \frac{-15.5}{4.77} \right] \approx P\{Z \geq -3.25\} \approx .9994 \)

(b) \( P\{59.5 \leq X \leq 70.5\} \approx P\{-5.5/4.77 \leq Z \leq 5.5/4.77\} = 2P\{Z \leq 1.15\} - 1 \approx .75 \)
(c) $P\{X \leq 74.5\} = P\{Z \leq 9.5/4.77\} = .977$

#5-8 (Theoretical Exercise)
Since $0 \leq X \leq c$, it follows that $X^2 \leq cX$. Hence,
\[
\begin{align*}
\text{Var}(X) &= E[X^2] - (E[X])^2 \\
&\leq E[cX] - (E[X])^2 \\
&= cE[X] - (E[X])^2 \\
&= E[X](c - E[X]) \\
&= c^2(\alpha(1-\alpha)) \text{ where } \alpha = E[X]/c \\
&\leq c^2/4
\end{align*}
\]
where the last inequality first uses the hypothesis that $P\{0 \leq X \leq c\} = 1$ to calculate that $0 \leq \alpha \leq 1$ and then uses calculus to show that $\max_{0 \leq \alpha \leq 1} \alpha(1-\alpha) = 1/4$.

2) a) mean is 1 and variance is 4/3.
   b) sample pdf and sample CDF approximate pdf and CDF with plots shown below.
   Sample mean is 1.0052 and sample variance is 1.3343.
   Matlab commands:
   ```matlab
dt=.002; t=-2:dt:4;
pdfu=.25*(t>= -1 & t<3);
subplot(2,2,1);
plot(t,pdfu)
axis([-2 4 0 .3])
ylabel('pdf of uniform [-1,3] RV')
cdfu=cumsum(pdfu)*(dt);
subplot(2,2,2)
plot(t,cdfu)
axis([-2 4 0 1.2])
ylabel('CDF of uniform [-1,3] RV')
u=4*rand(1,100000)-1;
[histu tu]= hist(u,100);
tu = [-1 tu 3];
histu= [0 histu/4000 0];
subplot(2,2,3)
plot(tu,histu)
axis([-2 4 0 0.3])
ylabel('sample pdf')
cdfus= cumsum(histu);
subplot(2,2,4)
plot(tu,cdfus/25)
axis([-1 3 0 1.2])
ylabel('sample CDF')
```

3) a) mean is 3 and variance is 4.
   b) sample pdf and sample CDF closely approximate pdf and CDF with plots shown below. Sample mean is 3.0037 and sample variance is 3.9822.
   Matlab commands:
   ```matlab
dt=.002; t=-3:dt:9; mg=3; vg=4;
pdfg= 1/sqrt(2*pi*vg)*exp(-.5*(t-3).^2/vg);
subplot(2,2,1);
```
```matlab
plot(t,pdg)
axis([-3 9 0 .3])
ylabel('pdf of Gaussian mean=3 std=2 RV')
cdfg=cumsum(pdg)*(dt);
subplot(2,2,2)
plot(t,cdfg)
axis([-3 9 0 1.2])
ylabel('CDF of Gaussian mean=3 std=2 RV')
g=2*randn(1,100000)+3;
[histg tg]= hist(g,100);
histg= histg/100000/(tg(2)-tg(1));
subplot(2,2,3)
plot(tg,histg)
axis([-3 9 0 0.3])
ylabel('sample pdf')
cdfgs= cumsum(histg);
subplot(2,2,4)
plot(tg,cdfgs*(tg(2)-tg(1)))
axis([-3 9 0 1.2])
ylabel('sample CDF')
```