EE342 Exam 2 Solutions

1) \[ f_X(x) = 2(e^{-x} - e^{-2x})u(x). \]

c) \[ F_X(x) = P(X \leq x) = \int_{-\infty}^{x} f_X(x) \, dx = (1 - 2e^{-x} + e^{-2x})u(x) = (1 - e^{-x})^2 u(x). \]

a) From c) \( P(1 < X < 2) = F_X(2) - F_X(1) = 2e^{-1} - 3e^{-2} + e^{-4}. \)

b) Note \( P(X > 2 | X > 1) = P(X > 2, X > 1) / P(X > 1) = P(X > 2) / P(X > 1) = (2e^{-2} - e^{-4}) / (2e^{-2} - e^{-4}) \).

d) Let \( U \) be a uniform \([0,1]\) random variable. Then using inverse distribution method \( u = (1 - e^{-x})^2 \) and \( x = -\log(1 - \sqrt{u}) \). The matlab commands are then given by
\[ x = -\log(1 - \text{sqrt}(	ext{rand}(1000,1))) \]

2)

a) Note \( p_{X,Y}(i,j) = p_X(i)p_{Y|X}(j|i) \). Probability trees can be used to calculate joint pmf giving
\[ p_{X,Y}(i,j) = \begin{cases} 1/5, & \text{if } i = 0, j = 0, \text{ or } i = 0, j = 1, \text{ or } i = 1, j = 0 \\ 1/15, & \text{if } i = 1, j = 1, \text{ or } i = 2, j = 1, \text{ or } i = 1, j = 2 \\ 1/10, & \text{if } i = 2, j = 0, \text{ or } i = 0, j = 2 \\ 0, & \text{otherwise} \end{cases} \]

b) Note that \( X \) and \( Y \) have the same pmf with \( p_X(i) = p_Y(i) = (1/2)\delta(i) + (1/3)\delta(i-1) + (1/6)\delta(i-2) \).

c) \( E(X) = E(Y) = 2/3, \ E(X^2) = E(Y^2) = 1, \ VAR(X) = VAR(Y) = 5/9, \ E(XY) = 1/3, \) and \( COV(X,Y) = -1/9. \)

d) Use the definition of conditional probability to get that \( p_{X|Y}(0|2) = 3/5 \) and \( p_{X|Y}(1|2) = 2/5. \)

3) \[ f_{X,Y} = \begin{cases} 8xy, & \text{if } 0 \leq y \leq x \leq 1, \\ 0, & \text{otherwise}, \end{cases} \]

a) \( f_X(x) = 4x^3(u(x) - u(x - 1)) \) and \( f_Y(y) = 4(y - y^3)(u(y) - u(y - 1)) \). \( X \) and \( Y \) are not independent as the joint pdf does not factor into the product of its marginals.

b) \( Z = X/Y \). Note that support of \( Z \), \( \mathbb{S}_Z = \{ z : z \geq 1 \} \).
\[ F_Z(z) = P(X \leq Yz) = \int_0^1 \int_{x/z}^x 8xydydx = (1 - 1/(z^2))u(z - 1) \]
Differentiating we get \( f_Z(z) = 2/(z^3)u(z - 1). \)

4)

a) Note \( E(X^2) = 8 \) and \( E(Y^2) = 25. \) If \( X \) and \( Y \) are independent, then \( E(Z) = -2, \ VAR(Z) = 13, \ E(W) = E(X)E(Y) = -8, \ E(W^2) = E(X^2)E(Y^2) = 200, \) and \( VAR(W) = 136 \). Note that \( E(WZ) = E(X^2Y) + E(XY^2) = -32 + 50 = 18 \) and \( \sigma_{Z,W} = 2 \). Therefore \( Z \) and \( W \) are correlated and therefore not independent.

b) \( VAR(Z) = VAR(X) + 2\sigma_{X,Y} + VAR(Y) = 4 + 2(-.5)(2)(3) + 9 = 7. \)