Description

A transportation problem basically deals with the problem, which aims to find the best way to fulfill the demand of n demand points using the capacities of m supply points. While trying to find the best way, generally a variable cost of shipping the product from one supply point to a demand point or a similar constraint should be taken into consideration.
7.1 Formulating Transportation Problems

Example 1: Powerco has three electric power plants that supply the electric needs of four cities.

• The associated supply of each plant and demand of each city is given in the table 1.
• The cost of sending 1 million kwh of electricity from a plant to a city depends on the distance the electricity must travel.

Transportation tableau

A transportation problem is specified by the supply, the demand, and the shipping costs. So the relevant data can be summarized in a transportation tableau. The transportation tableau implicitly expresses the supply and demand constraints and the shipping cost between each demand and supply point.
Table 1. Shipping costs, Supply, and Demand for Powerco Example

<table>
<thead>
<tr>
<th>From</th>
<th>To</th>
<th></th>
<th></th>
<th></th>
<th>Supply (Million kwh)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>City 1</td>
<td>City 2</td>
<td>City 3</td>
<td>City 4</td>
<td></td>
</tr>
<tr>
<td>Plant 1</td>
<td>$8</td>
<td>$6</td>
<td>$10</td>
<td>$9</td>
<td>35</td>
</tr>
<tr>
<td>Plant 2</td>
<td>$9</td>
<td>$12</td>
<td>$13</td>
<td>$7</td>
<td>50</td>
</tr>
<tr>
<td>Plant 3</td>
<td>$14</td>
<td>$9</td>
<td>$16</td>
<td>$5</td>
<td>40</td>
</tr>
<tr>
<td>Demand (Million kwh)</td>
<td>45</td>
<td>20</td>
<td>30</td>
<td>30</td>
<td></td>
</tr>
</tbody>
</table>

Solution

1. Decision Variable:

   Since we have to determine how much electricity is sent from each plant to each city;

   \[ X_{ij} = \text{Amount of electricity produced at plant } i \text{ and sent to city } j \]

   \[ X_{14} = \text{Amount of electricity produced at plant 1 and sent to city 4} \]
2. Objective function

Since we want to minimize the total cost of shipping from plants to cities;

Minimize $Z = 8X_{11} + 6X_{12} + 10X_{13} + 9X_{14}$
$+ 9X_{21} + 12X_{22} + 13X_{23} + 7X_{24}$
$+ 14X_{31} + 9X_{32} + 16X_{33} + 5X_{34}$

3. Supply Constraints

Since each supply point has a limited production capacity;

$X_{11} + X_{12} + X_{13} + X_{14} \leq 35$
$X_{21} + X_{22} + X_{23} + X_{24} \leq 50$
$X_{31} + X_{32} + X_{33} + X_{34} \leq 40$
4. Demand Constraints

Since each supply point has a limited production capacity;

\[ X_{11} + X_{21} + X_{31} \geq 45 \]
\[ X_{12} + X_{22} + X_{32} \geq 20 \]
\[ X_{13} + X_{23} + X_{33} \geq 30 \]
\[ X_{14} + X_{24} + X_{34} \geq 30 \]

5. Sign Constraints

Since a negative amount of electricity can not be shipped all Xij’s must be non negative;

\[ X_{ij} \geq 0 \quad (i = 1,2,3; \ j = 1,2,3,4) \]
LP Formulation of Powerco’s Problem

Min Z = 8X_{11}+6X_{12}+10X_{13}+9X_{14}+9X_{21}+12X_{22}+13X_{23}+7X_{24} \\
+14X_{31}+9X_{32}+16X_{33}+5X_{34}

S.T.: \begin{align*}
X_{11}+X_{12}+X_{13}+X_{14} & \leq 35 & \text{(Supply Constraints)} \\
X_{21}+X_{22}+X_{23}+X_{24} & \leq 50 \\
X_{31}+X_{32}+X_{33}+X_{34} & \leq 40 \\
X_{11}+X_{21}+X_{31} & \geq 45 & \text{(Demand Constraints)} \\
X_{12}+X_{22}+X_{32} & \geq 20 \\
X_{13}+X_{23}+X_{33} & \geq 30 \\
X_{14}+X_{24}+X_{34} & \geq 30 \\
X_{ij} & \geq 0 \text{ (i= 1,2,3; j= 1,2,3,4)} &
\end{align*}

General Description of a Transportation Problem

1. A set of \textit{m supply points} from which a good is shipped. Supply point \textit{i} can supply at most \textit{s_i} units.

2. A set of \textit{n demand points} to which the good is shipped. Demand point \textit{j} must receive at least \textit{d_i} units of the shipped good.

3. Each unit produced at supply point \textit{i} and shipped to demand point \textit{j} incurs a \textit{variable cost} of \textit{c_{ij}}.
$X_{ij} =$ number of units shipped from supply point $i$ to demand point $j$

$$
\text{min } \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} X_{ij}
$$

$$
ts.t. \sum_{j=1}^{n} X_{ij} \leq s_i (i = 1, 2, ..., m)
$$

$$
\sum_{i=1}^{m} X_{ij} \geq d_j (j = 1, 2, ..., n)
$$

$$
X_{ij} \geq 0 (i = 1, 2, ..., m; j = 1, 2, ..., n)
$$

Balanced Transportation Problem

If Total supply equals to total demand, the problem is said to be a balanced transportation problem:

$$
\sum_{i=1}^{m} S_i = \sum_{j=1}^{n} d_j
$$
Balancing a TP if total supply exceeds total demand

If total supply exceeds total demand, we can balance the problem by adding dummy demand point. Since shipments to the dummy demand point are not real, they are assigned a cost of zero.

Balancing a transportation problem if total supply is less than total demand

If a transportation problem has a total supply that is strictly less than total demand the problem has no feasible solution. There is no doubt that in such a case one or more of the demand will be left unmet. Generally in such situations a penalty cost is often associated with unmet demand and as one can guess this time the total penalty cost is desired to be minimum.
7.2 Finding Basic Feasible Solution for TP

Unlike other Linear Programming problems, a balanced TP with m supply points and n demand points is easier to solve, although it has m + n equality constraints. The reason for that is, if a set of decision variables (x$_{ij}$’s) satisfy all but one constraint, the values for x$_{ij}$’s will satisfy that remaining constraint automatically.

Methods to find the bfs for a balanced TP

There are three basic methods:

1. Northwest Corner Method
2. Minimum Cost Method
3. Vogel’s Method
1. Northwest Corner Method

To find the bfs by the NWC method:

Begin in the upper left (northwest) corner of the transportation tableau and set $x_{11}$ as large as possible (here the limitations for setting $x_{11}$ to a larger number, will be the demand of demand point 1 and the supply of supply point 1. Your $x_{11}$ value can not be greater than minimum of this 2 values).

According to the explanations in the previous slide we can set $x_{11}=3$ (meaning demand of demand point 1 is satisfied by supply point 1).
After we check the east and south cells, we saw that we can go east (meaning supply point 1 still has capacity to fulfill some demand).

After applying the same procedure, we saw that we can go south this time (meaning demand point 2 needs more supply by supply point 2).
Finally, we will have the following bfs, which is:
\[ x_{11} = 3, \ x_{12} = 2, \ x_{22} = 3, \ x_{23} = 2, \ x_{24} = 1, \ x_{34} = 2 \]

2. Minimum Cost Method

The Northwest Corner Method does not utilize shipping costs. It can yield an initial bfs easily but the total shipping cost may be very high. The minimum cost method uses shipping costs in order to come up with a bfs that has a lower cost. To begin the minimum cost method, first we find the decision variable with the smallest shipping cost \( X_{ij} \). Then assign \( X_{ij} \) its largest possible value, which is the minimum of \( s_i \) and \( d_j \).
After that, as in the Northwest Corner Method we should cross out row i and column j and reduce the supply or demand of the noncrossed-out row or column by the value of X_{ij}. Then we will choose the cell with the minimum cost of shipping from the cells that do not lie in a crossed-out row or column and we will repeat the procedure.

An example for Minimum Cost Method
Step 1: Select the cell with minimum cost.
**Step 2: Cross-out column 2**

<table>
<thead>
<tr>
<th></th>
<th>2</th>
<th>3</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

12 X 4 6

**Step 3: Find the new cell with minimum shipping cost and cross-out row 2**

<table>
<thead>
<tr>
<th></th>
<th>2</th>
<th>3</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

10 X 4 6
### Step 4: Find the new cell with minimum shipping cost and cross-out row 1

<table>
<thead>
<tr>
<th></th>
<th>2</th>
<th>3</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td></td>
<td></td>
<td></td>
<td>X</td>
</tr>
<tr>
<td>2</td>
<td>8</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>8</td>
<td>4</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>X</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### Step 5: Find the new cell with minimum shipping cost and cross-out column 1

<table>
<thead>
<tr>
<th></th>
<th>2</th>
<th>3</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td></td>
<td></td>
<td></td>
<td>X</td>
</tr>
<tr>
<td>2</td>
<td>8</td>
<td></td>
<td></td>
<td>X</td>
</tr>
<tr>
<td>3</td>
<td>8</td>
<td>4</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
<td>X</td>
<td>X</td>
<td></td>
</tr>
</tbody>
</table>
Step 6: Find the new cell with minimum shipping cost and cross-out column 3

Step 7: Finally assign 6 to last cell. The bfs is found as: $X_{11}=5$, $X_{21}=2$, $X_{22}=8$, $X_{31}=5$, $X_{33}=4$ and $X_{34}=6$
3. Vogel’s Method

Begin with computing each row and column a penalty. The penalty will be equal to the difference between the two smallest shipping costs in the row or column. Identify the row or column with the largest penalty. Find the first basic variable which has the smallest shipping cost in that row or column. Then assign the highest possible value to that variable, and cross-out the row or column as in the previous methods. Compute new penalties and use the same procedure.

An example for Vogel’s Method

Step 1: Compute the penalties.

<table>
<thead>
<tr>
<th>Supply</th>
<th>Row Penalty</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>7-6=1</td>
</tr>
<tr>
<td>15</td>
<td>78-15=63</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Demand</th>
<th>Column Penalty</th>
</tr>
</thead>
<tbody>
<tr>
<td>15</td>
<td>15-6=9</td>
</tr>
<tr>
<td>5</td>
<td>80-7=73</td>
</tr>
<tr>
<td>5</td>
<td>78-8=70</td>
</tr>
</tbody>
</table>
Step 2: Identify the largest penalty and assign the highest possible value to the variable.

```
<table>
<thead>
<tr>
<th>Supply</th>
<th>Row Penalty</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>5</td>
</tr>
<tr>
<td>15</td>
<td>80</td>
</tr>
<tr>
<td>8</td>
<td></td>
</tr>
<tr>
<td></td>
<td>8-6=2</td>
</tr>
<tr>
<td>15</td>
<td>78-15=63</td>
</tr>
<tr>
<td>Demand</td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>X</td>
</tr>
<tr>
<td>5</td>
<td></td>
</tr>
<tr>
<td>Column Penalty</td>
<td>15-6=9</td>
</tr>
<tr>
<td></td>
<td>_</td>
</tr>
<tr>
<td></td>
<td>78-8=70</td>
</tr>
</tbody>
</table>
```

Step 3: Identify the largest penalty and assign the highest possible value to the variable.

```
<table>
<thead>
<tr>
<th>Supply</th>
<th>Row Penalty</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>5</td>
</tr>
<tr>
<td>15</td>
<td>80</td>
</tr>
<tr>
<td>8</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>_</td>
</tr>
<tr>
<td></td>
<td>15</td>
</tr>
<tr>
<td></td>
<td>_</td>
</tr>
</tbody>
</table>
```

```
| Demand |             |
| 15     | X           |
|        | X           |
| Column Penalty | 15-6=9 |
|           | _           |
|           | _           |
```
Step 4: Identify the largest penalty and assign the highest possible value to the variable.

<table>
<thead>
<tr>
<th></th>
<th>Supply</th>
<th>Row Penalty</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 5 5 8</td>
<td>X 15</td>
<td></td>
</tr>
</tbody>
</table>

Demand: 15 X X

Column Penalty: __ __ __

Step 5: Finally the bfs is found as $X_{11} = 0$, $X_{12} = 5$, $X_{13} = 5$, and $X_{21} = 15$.

<table>
<thead>
<tr>
<th></th>
<th>Supply</th>
<th>Row Penalty</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 5 5 8</td>
<td>X 15</td>
<td></td>
</tr>
</tbody>
</table>

Demand: X X X

Column Penalty: __ __ __
7.3 The Transportation Simplex Method

In this section we will explain how the simplex algorithm is used to solve a transportation problem.

How to Pivot a Transportation Problem

Based on the transportation tableau, the following steps should be performed.

Step 1. Determine (by a criterion to be developed shortly, for example northwest corner method) the variable that should enter the basis.

Step 2. Find the loop (it can be shown that there is only one loop) involving the entering variable and some of the basic variables.

Step 3. Counting the cells in the loop, label them as even cells or odd cells.
Step 4. Find the odd cells whose variable assumes the smallest value. Call this value \( \theta \). The variable corresponding to this odd cell will leave the basis. To perform the pivot, decrease the value of each odd cell by \( \theta \) and increase the value of each even cell by \( \theta \). The variables that are not in the loop remain unchanged. The pivot is now complete. If \( \theta = 0 \), the entering variable will equal 0, and an odd variable that has a current value of 0 will leave the basis. In this case a degenerate bfs existed before and will result after the pivot. If more than one odd cell in the loop equals \( \theta \), you may arbitrarily choose one of these odd cells to leave the basis; again a degenerate bfs will result.

Illustration of pivoting procedure on the Powerco example. We want to find the bfs that would result if \( X_{14} \) were entered into the basis.
New bfs after $X_{14}$ is pivoted into basis. Since there is no loop involving the cells (1,1), (1,4), (2,1), (2,2), (3,3) and (3,4) the new solution is a bfs.

<table>
<thead>
<tr>
<th></th>
<th>35-20</th>
<th></th>
<th>0+20</th>
<th>35</th>
</tr>
</thead>
<tbody>
<tr>
<td>10+20</td>
<td>20</td>
<td>20-20</td>
<td>(nonbasic)</td>
<td>50</td>
</tr>
<tr>
<td></td>
<td>10+20</td>
<td>30-20</td>
<td>40</td>
<td></td>
</tr>
<tr>
<td>45</td>
<td>20</td>
<td>30</td>
<td>30</td>
<td></td>
</tr>
</tbody>
</table>

After the pivot the new bfs is $X_{11}=15$, $X_{14}=20$, $X_{21}=30$, $X_{22}=20$, $X_{33}=30$ and $X_{34}=10$.

Two important points!

In the pivoting procedure:

1. Since each row has as many $+20$s as $-20$s, the new solution will satisfy each supply and demand constraint.

2. By choosing the smallest odd variable ($X_{23}$) to leave the basis, we ensured that all variables will remain nonnegative.
Pricing out nonbasic variables

To complete the transportation simplex, now we will discuss how to row 0 for any bfs. For a bfs in which the set of basic variables is BV, the coefficient of the variable $X_{ij}$ (call it $\hat{c}_{ij}$) in the tableau’s row is given by

$$\hat{c}_{ij} = c_{BV}B^{-1}a_{ij} - c_{ij}$$

Where $c_{ij}$ is the objective function coefficient for $X_{ij}$ and $a_{ij}$ is the column for $x_{ij}$ in the original LP.

Since the example is a minimization problem, the current bfs will be optimal if all the $\hat{c}_{ij}$’s are nonpositive; otherwise, we enter into the basis with the most positive $\hat{c}_{ij}$.

After determining $c_{BV}B^{-1}$ we can easily determine $\hat{c}_{ij}$. Since the first constraint has been dropped, $c_{BV}B^{-1}$ will have $m+n-1$ elements.

$$c_{BV}B^{-1} = [u_2 \ u_3 \ldots u_m \ v_1 \ v_2 \ldots v_n]$$

Where $u_2, u_3, \ldots u_m$ are elements of $c_{BV}B^{-1}$ corresponding to the $m-1$ supply constraints, and $v_1, v_2, \ldots v_n$ are elements of $c_{BV}B^{-1}$ corresponding to the $n$ demand constraints.
To determine $c_{BV}B^{-1}$ we use the fact that in any tableau, each basic variable $X_{ij}$ must have $\tilde{c}_{ij}=0$. Thus for each of the $m+n-1$ variables in BV,

$$c_{BV}B^{-1}a_{ij} - c_{ij}=0$$

For the Northwest corner bfs of Powerco problem, $BV=\{X_{11}, X_{21}, X_{22}, X_{23}, X_{33}, X_{34}\}$. Applying the equation above we obtain:

$$\tilde{c}_{11}=\begin{bmatrix} u_2 & u_3 & v_1 & v_2 & v_3 & v_4 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ -8 \\ 0 \\ 0 \end{bmatrix} = v_1-8=0$$

$$\tilde{c}_{21}=\begin{bmatrix} u_2 & u_3 & v_1 & v_2 & v_3 & v_4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = u_2+v_1-9=0$$

$$\tilde{c}_{22}=\begin{bmatrix} u_2 & u_3 & v_1 & v_2 & v_3 & v_4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = u_2+v_2-12=0$$

$$\tilde{c}_{23}=\begin{bmatrix} u_2 & u_3 & v_1 & v_2 & v_3 & v_4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = u_2+v_3-13=0$$
\[
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
u_1 \\
u_2 \\
u_3 \\
u_4
\end{bmatrix} - 16 = u_3 + v_3 - 16 = 0
\]

\[
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
u_1 \\
u_2 \\
u_3 \\
u_4
\end{bmatrix} - 5 = u_3 + v_4 - 5 = 0
\]

For each basic variable \(X_{ij}\) (except those having \(i=1\)), we see that the equation we used above reduces to \(u_i + v_j = c_{ij}\). If we define \(u_1 = 0\), we must solve the following system of \(m+n\) equations.

\[
\begin{align*}
u_1 &= 0 \\
u_1 + v_1 &= 8 \\
u_2 + v_1 &= 9 \\
u_2 + v_2 &= 12 \\
u_2 + v_3 &= 13 \\
u_3 + v_3 &= 16 \\
u_3 + v_4 &= 5
\end{align*}
\]

By solving the system above we obtain:

\[
\begin{align*}
u_1 &= 8, \quad u_2 = 1, \quad v_2 = 11, \quad v_3 = 12, \quad u_3 = 4, \quad v_4 = 1
\end{align*}
\]
For each nonbasic variable, we now compute 
\[ \hat{c}_{ij} = u_i + v_j - c_{ij} \]

We obtain:
\[ \hat{c}_{12} = 0 + 11 - 6 = 5 \]
\[ \hat{c}_{13} = 0 + 12 - 10 = 2 \]
\[ \hat{c}_{14} = 0 + 1 - 9 = -8 \]
\[ \hat{c}_{24} = 1 + 1 - 7 = -5 \]
\[ \hat{c}_{31} = 4 + 8 - 14 = -2 \]
\[ \hat{c}_{32} = 4 + 11 - 9 = 6 \]

Since \( \hat{c}_{32} \) is the greatest positive \( \hat{c}_{ij} \), we would next enter \( X_{32} \) into the basis. Each unit that is entered into the basis will decrease Powerco’s cost by $6.

We have determined that \( X_{32} \) should enter the basis. As shown in the table below the loop involving \( X_{32} \) and some of the basic variables is (3,2), (3,3), (2,3), (2,2). The odd cells in the loop are (2,2) and (3,3). Since the smallest value of these two is 10 the pivot is 10.
The resulting bfs will be:
\[ X_{11}=35, \ X_{32}=10, \ X_{21}=10, \ X_{22}=10, \ X_{23}=30 \text{ and } X_{34}=30 \]

The \( u_i \)'s and \( v_j \)'s for the new bfs were obtained by solving:
\[
\begin{align*}
    u_1 &= 0 \\
    u_2 + v_2 &= 12 \\
    u_3 + v_4 &= 5 \\
    u_1 + v_1 &= 8 \\
    u_2 + v_3 &= 13 \\
    u_2 + v_1 &= 9 \\
    u_3 + v_2 &= 9
\end{align*}
\]

In computing \( \hat{c}_{ij} = u_i + v_j - c_{ij} \) for each nonbasic variable, we find that \( \hat{c}_{12} = 5, \ \hat{c}_{13} = 2 \) and \( \hat{c}_{24} = 1 \) are the only positive \( \hat{c}_{ij} \)'s. Thus we next enter \( X_{12} \) into the basis. By applying the same steps we will finally get a solution where all \( \hat{c}_{ij} \)'s are less than or equal to 0, so an optimal solution has been obtained.

The optimal solution for Powerco is \( X_{11}=10, \ X_{13}=25, \ X_{21}=45, \ X_{23}=5, \ X_{32}=10 \) and \( X_{34}=30 \).

As a result of this solution the objective function value becomes:
\[
Z=6(10)+10(25)+9(45)+13(5)+9(10)+5(30)=1020
\]
7.4 Sensitivity Analysis

In this section we discuss the following three aspects of sensitivity analysis for the transportation problem:
1. Changing the objective function coefficient of a nonbasic variable.
2. Changing the objective function coefficient of a basic variable.
3. Increasing a single supply by $\Delta$ and a single demand by $\Delta$.

1. Changing the objective function coefficient of a nonbasic variable.

Changing the objective function coefficient of a nonbasic variable $X_{ij}$ will leave the right hand side of the optimal tableau unchanged. Thus the current basis will still be feasible. Since we are not changing $c_{B^*}B^{-1}$, the $u_i$’s and $v_j$’s remain unchanged. In row 0 only the coefficient of $X_{ij}$ will change. Thus as long as the coefficient of $X_{ij}$ in the optimal row 0 is nonpositive, the current basis remains optimal.
Let’s try to answer the following question about Powerco as an example:

For what range of values of the cost of shipping 1 million kwh of electricity from plant 1 to city 1 will the current basis remain optimal?

Suppose we change $c_{11}$ from 8 to $8 + \Delta$.

Now $\bar{c}_{11} = u_1 + v_1 - c_{11} = 0 + 6 - (8 + \Delta) = 2 - \Delta$.

Thus the current basis remains optimal for $-2 - \Delta \leq 0$, or $\Delta \geq -2$, and $c_{11} \geq 8 - 2 = 6$.

2. Changing the objective function coefficient of a basic variable.

Since we are changing $c_{B'V'}B^{-1}$, the coefficient of each nonbasic variable in row 0 may change, and to determine whether the current basis remain optimal, we must find the new $u_i$’s and $v_j$’s and use these values to price out all nonbasic variables. The current basis remains optimal as long as all nonbasic variables price out nonpositive.
Let’s try to answer the following question about Powerco as an example:

For what range of values of the cost of shipping 1 million kwh of electricity from plant 1 to city 3 will the current basis remain optimal?

Suppose we change $c_{13}$ from 10 to $10 + \Delta$.

Now $\check{c}_{13} = 0$ changes from $u_1 + v_3 = 10$ to $u_1 + v_3 = 10 + \Delta$.

Thus, to find the $u_i$’s and $v_j$’s we must solve the following equations:

\[
\begin{align*}
  u_1 &= 0 \\
  u_1 + v_2 &= 6 \\
  u_2 + v_1 &= 9 \\
  u_2 + v_3 &= 13 \\
  u_3 + v_2 &= 9 \\
  u_1 + v_3 &= 10 + \Delta \\
  u_3 + v_4 &= 5
\end{align*}
\]

Solving these equations, we obtain $u_1 = 0$, $v_2 = 6$, $v_3 = 10 + \Delta$, $v_1 = 6 + \Delta$, $u_2 = 3 - \Delta$, $u_3 = 3$, and $v_4 = 2$.

We now price out each nonbasic variable. The current basis will remain optimal as long as each nonbasic variable has a nonpositive coefficient in row 0.

\[
\begin{align*}
  \check{c}_{11} &= u_1 + v_1 - 8 = \Delta - 2 \leq 0 \quad \text{for } \Delta \leq 2 \\
  \check{c}_{14} &= u_1 + v_4 - 9 = -7 \\
  \check{c}_{22} &= u_2 + v_2 - 12 = -3 - \Delta \leq 0 \quad \text{for } \Delta \geq -3 \\
  \check{c}_{24} &= u_2 + v_4 - 7 = -2 - \Delta \leq 0 \quad \text{for } \Delta \geq -2 \\
  \check{c}_{31} &= u_3 + v_1 - 14 = -5 + \Delta \leq 0 \quad \text{for } \Delta \leq 5 \\
  \check{c}_{33} &= u_3 + v_3 - 16 = -3 \leq 0 \quad \text{for } \Delta \leq 3
\end{align*}
\]
Thus, the current basis remains optimal for $-2 \leq \Delta \leq 2$,
or $8 = 10 - 2 \leq c_{ij} \leq 10 + 2 = 12$

3. Increasing Both Supply $s_i$ and Demand $d_j$ by $\Delta$.

Changing both supply and demand by the same amount will maintain the balance of the transportation problem. Since $u_i$’s and $v_j$’s may be thought of as the negative of each constraint’s shadow price, we know that if the current basis remains optimal,

New $Z$ value = old $Z$ value $+ \Delta u_i + \Delta v_j$

For example if we increase plant 1’s supply and city 2’s demand by 1 unit, then

New cost = $1020 + 1(0) + 1(6) = $1026
We can also find the new values of the decision variables as follows:

1. If $X_{ij}$ is a basic variable in the optimal solution, increase $X_{ij}$ by $\Delta$.

2. If $X_{ij}$ is a nonbasic variable in the optimal solution, find the loop involving $X_{ij}$ and some of the basic variables. Find an odd cell in the loop that is in row $I$. Increase the value of this odd cell by $\Delta$ and go around the loop, alternately increasing and then decreasing current basic variables in the loop by $\Delta$.

7.5. Assignment Problems

Example: Machineco has four jobs to be completed. Each machine must be assigned to complete one job. The time required to setup each machine for completing each job is shown in the table below. Machinco wants to minimize the total setup time needed to complete the four jobs.
Setup times
(Also called the cost matrix)

<table>
<thead>
<tr>
<th></th>
<th>Time (Hours)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Job1</td>
</tr>
<tr>
<td>Machine 1</td>
<td>14</td>
</tr>
<tr>
<td>Machine 2</td>
<td>2</td>
</tr>
<tr>
<td>Machine 3</td>
<td>7</td>
</tr>
<tr>
<td>Machine 4</td>
<td>2</td>
</tr>
</tbody>
</table>

The Model

According to the setup table Machinco’s problem can be formulated as follows (for $i,j=1,2,3,4$):

$$
\begin{align*}
\min Z &= 14X_{11} + 5X_{12} + 8X_{13} + 7X_{14} + 2X_{21} + 12X_{22} + 6X_{23} + 5X_{24} \\
&+ 7X_{31} + 8X_{32} + 3X_{33} + 9X_{34} + 2X_{41} + X_{42} + 6X_{43} + 10X_{44} \\
\text{s.t.} & X_{11} + X_{12} + X_{13} + X_{14} = 1 \\
& X_{21} + X_{22} + X_{23} + X_{24} = 1 \\
& X_{31} + X_{32} + X_{33} + X_{34} = 1 \\
& X_{41} + X_{42} + X_{43} + X_{44} = 1 \\
& X_{11} + X_{21} + X_{31} + X_{41} = 1 \\
& X_{12} + X_{22} + X_{32} + X_{42} = 1 \\
& X_{13} + X_{23} + X_{33} + X_{43} = 1 \\
& X_{14} + X_{24} + X_{34} + X_{44} = 1 \\
& X_0 = 0 \text{or} X_1 = 1
\end{align*}
$$
For the model on the previous page note that:

\[ X_{ij} = 1 \text{ if machine } i \text{ is assigned to meet the demands of job } j \]

\[ X_{ij} = 0 \text{ if machine } i \text{ is assigned to meet the demands of job } j \]

In general an assignment problem is balanced transportation problem in which all supplies and demands are equal to 1.

Although the transportation simplex appears to be very efficient, there is a certain class of transportation problems, called assignment problems, for which the transportation simplex is often very inefficient. For that reason there is another method called The Hungarian Method. The steps of The Hungarian Method are as listed below:

**Step 1.** Find a bfs. Find the minimum element in each row of the \( m \times m \) cost matrix. Construct a new matrix by subtracting from each cost the minimum cost in its row. For this new matrix, find the minimum cost in each column. Construct a new matrix (reduced cost matrix) by subtracting from each cost the minimum cost in its column.
**Step 2.** Draw the minimum number of lines (horizontal and/or vertical) that are needed to cover all zeros in the reduced cost matrix. If \( m \) lines are required, an optimal solution is available among the covered zeros in the matrix. If fewer than \( m \) lines are required, proceed to step 3.

**Step 3.** Find the smallest nonzero element (call its value \( k \)) in the reduced cost matrix that is uncovered by the lines drawn in step 2. Now subtract \( k \) from each uncovered element of the reduced cost matrix and add \( k \) to each element that is covered by two lines. Return to step 2.

### 7.6 Transshipment Problems

A transportation problem allows only shipments that go directly from supply points to demand points. In many situations, shipments are allowed between supply points or between demand points. Sometimes there may also be points (called transshipment points) through which goods can be transshipped on their journey from a supply point to a demand point. Fortunately, the optimal solution to a transshipment problem can be found by solving a transportation problem.
The following steps describe how the optimal solution to a transshipment problem can be found by solving a transportation problem.

Step 1. If necessary, add a dummy demand point (with a supply of 0 and a demand equal to the problem’s excess supply) to balance the problem. Shipments to the dummy and from a point to itself will be zero. Let \( s \) = total available supply.

Step 2. Construct a transportation tableau as follows: A row in the tableau will be needed for each supply point and transshipment point, and a column will be needed for each demand point and transshipment point.

Each supply point will have a supply equal to its original supply, and each demand point will have a demand to its original demand. Let \( s \) = total available supply. Then each transshipment point will have a supply equal to (point’s original supply) + \( s \) and a demand equal to (point’s original demand) + \( s \). This ensures that any transshipment point that is a net supplier will have a net outflow equal to point’s original supply and a net demander will have a net inflow equal to point’s original demand. Although we don’t know how much will be shipped through each transshipment point, we can be sure that the total amount will not exceed \( s \).