Chapter 3
Introduction to Linear Programming

to accompany
Introduction to Mathematical Programming: Operations Research, Volume 1
4th edition, by Wayne L. Winston and Munirpallam Venkataramanan

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3.1 - What is a Linear Programming Problem?

Example

Giapetto’s, Inc., manufactures wooden soldiers and trains.

Each soldier built:
- Sell for $27 and uses $19 worth of raw materials.
- Increase Giapetto’s variable labor/overhead costs by $14.
- Requires 2 hours of finishing labor.
- Requires 1 hour of carpentry labor.

Each train built:
- Sell for $21 and used $9 worth of raw materials.
- Increases Giapetto’s variable labor/overhead costs by $10.
- Requires 1 hour of finishing labor.
- Requires 1 hour of carpentry labor.
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Each week Giapetto can obtain:

- All needed raw material.
- Only 100 finishing hours.
- Only 80 carpentry hours.

Also:

- Demand for the trains is unlimited.
- At most 40 soldiers are bought each week.

Giapetto wants to maximize weekly profit (revenues – expenses). Formulate a mathematical model of Giapetto’s situation that can be used to maximize weekly profit.

The Giapetto solution model incorporates the characteristics shared by all linear programming problems.

**Decision Variables**

- \( x_1 = \) number of soldiers produced each week
- \( x_2 = \) number of trains produced each week

**Objective Function**

In any linear programming model, the decision maker wants to maximize (usually revenue or profit) or minimize (usually costs) some function of the decision variables. This function, to be maximized or minimized, is called the objective function. For the Giapetto problem, fixed costs are do not depend upon the values of \( x_1 \) or \( x_2 \).
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Giapetto’s weekly profit can be expressed in terms of the decision variables $x_1$ and $x_2$:

Weekly profit =
weekly revenue – weekly raw material costs – the weekly variable costs

Weekly revenue = $27x_1 + 21x_2$  
Weekly raw material costs = $10x_1 + 9x_2$  
Weekly variable costs = $14x_1 + 10x_2$

Weekly profit =  
$(27x_1 + 21x_2) - (10x_1 + 9x_2) - (14x_1 + 10x_2) = 3x_1 + 2x_2$

Thus, Giapetto’s objective is to choose $x_1$ and $x_2$ to maximize $3x_1 + 2x_2$. We use the variable $z$ to denote the objective function value of any LP. Giapetto’s objective function is:

**Maximize** $z = 3x_1 + 2x_2$

“Maximize” will be abbreviated by *max* and “minimize” by *min*. The coefficient of an objective function variable is called an objective function coefficient.
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Constraints As x₁ and x₂ increase, Giapetto’s objective function grows larger. For Giapetto, the values of x₁ and x₂ are limited by the following three restrictions (often called constraints):

Constraint 1  Each week, no more than 100 hours of finishing time may be used.
Constraint 2  Each week, no more than 80 hours of carpentry time may be used.
Constraint 3  Because of limited demand, at most 40 soldiers should be produced.

These three constraints can be expressed mathematically by the following equations:

Constraint 1: \[ 2x_1 + x_2 \leq 100 \]
Constraint 2: \[ x_1 + x_2 \leq 80 \]
Constraint 3: \[ x_1 \leq 40 \]

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The coefficients of the constraints are often called the technological coefficients. The number on the right-hand side of the constraint is called the constraint’s right-hand side (or rhs).

Sign Restrictions To complete the formulation of a linear programming problem, the following question must be answered for each decision variable: Can the decision variable only assume nonnegative values, or is the decision variable allowed to assume both positive and negative values?

If the decision variable can assume only nonnegative values, the sign restriction \( x_i \geq 0 \) is added. If the variable can assume both positive and negative values, the decision variable \( x_i \) is unrestricted in sign (often abbreviated urs).
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For the Giapetto problem model, combining the sign restrictions \( x_1 \geq 0 \) and \( x_2 \geq 0 \) with the objective function and constraints yields the following optimization model:

Max \( z = 3x_1 + 2x_2 \) (objective function)

Subject to (s.t.)

\[
\begin{align*}
2x_1 + x_2 &\leq 100 \quad \text{(finishing constraint)} \\
x_1 + x_2 &\leq 80 \quad \text{(carpentry constraint)} \\
x_1 &\leq 40 \quad \text{(constraint on demand for soldiers)} \\
x_1 &\geq 0 \quad \text{(sign restriction)} \\
x_2 &\geq 0 \quad \text{(sign restriction)}
\end{align*}
\]

Concepts of linear function and linear inequality:

**Linear Function**: A function \( f(x_1, x_2, \ldots, x_n) \) of \( x_1, x_2, \ldots, x_n \) is a linear function if and only if for some set of constants, \( c_1, c_2, \ldots, c_n \), \( f(x_1, x_2, \ldots, x_n) = c_1x_1 + c_2x_2 + \ldots + c_nx_n \).

For example, \( f(x_1, x_2) = 2x_1 + x_2 \) is a linear function of \( x_1 \) and \( x_2 \), but \( f(x_1, x_2) = (x_1)^2x_2 \) is not a linear function of \( x_1 \) and \( x_2 \).

For any linear function \( f(x_1, x_2, \ldots, x_n) \) and any number \( b \), the inequalities inequality \( f(x_1, x_2, \ldots, x_n) \leq b \) and \( f(x_1, x_2, \ldots, x_n) \geq b \) are linear inequalities.
A linear programming problem (LP) is an optimization problem for which we do the following:

1. Attempt to maximize (or minimize) a linear function (called the objective function) of the decision variables.
2. The values of the decision variables must satisfy a set of constraints. Each constraint must be a linear equation or inequality.
3. A sign restriction is associated with each variable. For each variable $x_i$, the sign restriction specifies either that $x_i$ must be nonnegative ($x_i \geq 0$) or that $x_i$ may be unrestricted in sign.

Proportionality and Additive Assumptions

The fact that the objective function for an LP must be a linear function of the decision variables has two implications:

1. The contribution of the objective function from each decision variable is proportional to the value of the decision variable. For example, the contribution to the objective function for 4 soldiers is exactly four times the contribution of 1 soldier.
2. The contribution to the objective function for any variable is independent of the other decision variables. For example, no matter what the value of $x_2$, the manufacture of $x_1$ soldiers will always contribute $3x_1$ dollars to the objective function.
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Analogously, the fact that each LP constraint must be a linear inequality or linear equation has two implications:

1. The contribution of each variable to the left-hand side of each constraint is proportional to the value of the variable. For example, it takes exactly 3 times as many finishing hours to manufacture 3 soldiers as it does 1 soldier.

2. The contribution of a variable to the left-hand side of each constraint is independent of the values of the variable. For example, no matter what the value of $x_1$, the manufacture of $x_2$ trains uses $x_2$ finishing hours and $x_2$ carpentry hours.

Divisibility Assumption

The divisibility assumption requires that each decision variable be permitted to assume fractional values. For example, this assumption implies it is acceptable to produce a fractional number of trains. The Giapetto LP does not satisfy the divisibility assumption since a fractional soldier or train cannot be produced. Chapter 9 the use of integer programming methods necessary to address the solution to this problem.

The Certainty Assumption

The certainty assumption is that each parameter (objective function coefficients, right-hand side, and technological coefficients) are known with certainty.
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Feasible Region and Optimal Solution

The feasible region of an LP is the set of all points satisfying all the LP’s constraints and sign restrictions.

$x_1 = 40$ and $x_2 = 20$ are in the feasible region since they satisfy all the Giapetto constraints.

On the other hand, $x_1 = 15$, $x_2 = 70$ is not in the feasible region because this point does not satisfy the carpentry constraint [15 + 70 is > 80].

Giapetto Constraints

2 $x_1 + x_2 \leq 100$ (finishing constraint)

$x_1 + x_2 \leq 80$ (carpentry constraint)

$x_1 \leq 40$ (demand constraint)

$x_1 \geq 0$ (sign restriction)

$x_2 \geq 0$ (sign restriction)

3.1 - What Is a Linear Programming Problem?

For a maximization problem, an optimal solution to an LP is a point in the feasible region with the largest objective function value. Similarly, for a minimization problem, an optimal solution is a point in the feasible region with the smallest objective function value.

Most LPs have only one optimal solution. However, some LPs have no optimal solution, and some LPs have an infinite number of solutions. Section 3.2 shows the optimal solution to the Giapetto LP is $x_1 = 20$ and $x_2 = 60$. This solution yields an objective function value of:

$$z = 3x_1 + 2x_2 = 3(20) + 2(60) = 180$$

When we say $x_1 = 20$ and $x_2 = 60$ is the optimal solution, we are saying that no point in the feasible region has an objective function value (profit) exceeding 180.
Any LP with only two variables can be solved graphically. We always label the variables $x_1$ and $x_2$ and the coordinate axes the $x_1$ and $x_2$ axes.

Graphical Example:
The shaded area in the graph are the set of points satisfying the equation:
$2x_1 + 3x_2 \leq 6$

Finding the Feasible Solution
Since the Giapetto LP has two variables, it may be solved graphically. The feasible region is the set of all points satisfying the constraints:

\[
\begin{align*}
2x_1 + x_2 & \leq 100 \quad \text{finishing constraint} \\
x_1 + x_2 & \leq 80 \quad \text{carpentry constraint} \\
x_1 & \leq 40 \quad \text{demand constraint} \\
x_1 & \geq 0 \quad \text{sign restriction} \\
x_2 & \geq 0 \quad \text{sign restriction}
\end{align*}
\]

A graph of the constraints and feasible region is shown on the next slide.
From figure, we see that the set of points satisfying the Giapetto LP is bounded by the five sided polygon DGFEH. Any point on or in the interior of this polygon (the shade area) is in the feasible region.

Having identified the feasible region for the Giapetto LP, a search can begin for the optimal solution which will be the point in the feasible region with the largest z-value.
To find the optimal solution, graph a line on which the points have the same z-value. In a max problem, such a line is called an **isoprofit** line while in a min problem, this is called the **isocost** line. The figure shows the isoprofit lines for \( z = 60, \ z = 100, \) and \( z = 180 \)

The last isoprofit intersecting (touching) the feasible region indicates the optimal solution for the LP. For the Giapetto problem, this occurs at point G \( (x_1 = 20, \ x_2 = 60, \ z = 180) \).
3.2 – Graphical Solution to a 2-Variable LP

Binding and Nonbinding constraints

Once the optimal solution to an LP is found, it is useful to classify each constraint as being a binding or nonbinding constraint.

A constraint is **binding** if the left-hand and right-hand side of the constraint are **equal** when the optimal values of the decision variables are substituted into the constraint. In the Giapetto LP, the finishing and carpentry constraints are binding.

A constraint is **nonbinding** if the left-hand side and the right-hand side of the constraint are **unequal** when the optimal values of the decision variables are substituted into the constraint. In the Giapetto LP, the demand constraint for wooden soldiers is nonbinding since at the optimal solution ($x_1 = 20$), $x_1 < 40$. 
A set of points $S$ is a **convex** set if the line segment jointing any two pairs of points in $S$ is wholly contained in $S$.

For any convex set $S$, a point $p$ in $S$ is an **extreme point** if each line segment that lines completely in $S$ and contains the point $P$ has $P$ as an endpoint of the line segment.

Consider the figures (a) – (d) below:

For example, in figures (a) and (b) below, each line segment joining points in $S$ contains only points in $S$. Thus is convex for (a) and (b). In both figures (c) and (d), there are points in the line segment $AB$ that are not in $S$. $S$ is not convex for (c) and (d).
In figure (a), each point on the circumference of the circle is an extreme point of the circle. In figure (b), A, B, C, and D are extreme points of S. Point E is not an extreme point since E is not an end point of the line segment AB.

Extreme points are sometimes called corner points, because if the set S is a polygon, the extreme points will be the vertices, or corners, of the polygon.

The feasible region for the Giapetto LP will be a convex set.
It can be shown that:

• The feasible region for any LP will be a convex set.

• The feasible region for any LP has only a finite number of extreme points.

1. Any LP that has an optimal solution has an extreme point that is optimal.

For the Giapetto problem, the optimal solution (Point G) must be any extreme point of the feasible region. Since $z$ increases as we move isoprofit lines in a northeast direction, the largest $z$ in the feasible region occurs at some point P that has no points in the feasible region northeast of P.
This means the optimal solution must lie somewhere on the boundary of the feasible region. The LP must have an extreme point that is optimal, because for any line segment on the boundary of the feasible area, the largest z value on that line segment must be assumed at be at one endpoints of the line segment.

3.2 – Graphical Solution to a 2-Variable LP

A Graphical Solution to a Minimization Problem

Dorian Auto manufactures luxury cars and trucks. The company believes that its most likely customers are high-income women and men.

To reach these groups, Dorian Auto has embarked on an ambitious TV advertising campaign and has decide to purchase 1-minute commercial spots on two type of programs: comedy shoes and football games.
3.2 – Graphical Solution to a 2-Variable LP

• Each comedy commercial is seen by 7 million high income women and 2 million high-income men.
• Each football game is seen by 2 million high-income women and 12 million high-income men.
• A 1-minute comedy ad costs $50,000 and a 1-minute football ad costs $100,000.

Dorian Auto would like for commercials to be seen by at least 28 million high-income women and 24 million high-income men.

Use LP to determine how Dorian Auto can meet its advertising requirements at minimum cost.

Problem Formulation

The decision variables are: 
\[ x_1 = \text{number of 1-minute comedy ads} \]
\[ x_2 = \text{number of 1-minute football ads} \]

Min \[ z = 50x_1 + 100x_2 \] \hspace{1cm} \text{(objective function in $1,000)}

s.t. \[ 7x_1 + 2x_2 \geq 28 \] \hspace{1cm} \text{(high-income women)}
\[ 2x_1 + 12x_2 \geq 24 \] \hspace{1cm} \text{(high-income men)}
\[ x_1, x_2 \geq 0 \] \hspace{1cm} \text{(non-negativity constraints)}

A graph of the feasible area is shown on the next slide.
3.2 – Graphical Solution to a 2-Variable LP

Like the Giapetto LP, the Dorian LP has a convex feasible region. The feasible region for the Dorian problem, however, contains points for which the value of at least one variable can assume arbitrarily large values. Such a feasible region is called an unbounded feasible region.

Since Dorian wants to minimize total advertising costs, the optimal solution to the problem is the point in the feasible region with the smallest z value. An isocost line with the smallest z value passes through point E and is the optimal solution at \( x_1 = 3.4 \) and \( x_2 = 1.4 \).
3.2 – Graphical Solution to a 2-Variable LP

Because at point E, both the high-income women and high-income men constraints are satisfied, both constraints are binding. The Additivity Assumption was used in writing: total viewers = comedy viewer ads + football ad viewers. Since many of the same people might view both ads, double-counting of such people would occur thereby violating the Additivity Assumption.

If only 1-minute commercials are available, it is unreasonable to say 3.6 comedy and 1.4 football commercials should be purchased. So, the Divisibility Assumption has been violated, and the Dorian LP should, in reality, be considered as an integer programming problem (Chapter 9). Since there is no way of knowing with certainty how many viewers are added with each type of commercial, the Certainty Assumption is also violated.

Despite these violations, analysts have used similar models to help companies determine their optimal media mix.
Some LPs have an infinite number of solutions. Consider the following formulation:

$$\text{max } z = 3x_1 + 2x_2$$

s.t. $\frac{1}{40}x_1 + \frac{1}{60}x_2 \leq 1$

$\frac{1}{50}x_1 + \frac{1}{50}x_2 \leq 1$

$x_1, x_2 \geq 0$

Any point (solution) falling on line segment $AE$ will yield an optimal solution of $z = 120$. 

Some LPs have an infinite number of solutions (alternative or multiple optimal solutions).

Some LPs have no feasible solutions (infeasible LPs).

Some LPs are unbounded: There are points in the feasible region with arbitrarily large (in a maximization problem) $z$-values.
Some LPs have no solution. Consider the following formulation:

\[ \text{max } z = 3x_1 + 2x_2 \]
\[ \text{s.t. } \begin{align*}
\frac{1}{40}x_1 + \frac{1}{60}x_2 & \leq 1 \\
\frac{1}{50}x_1 + \frac{1}{50}x_2 & \leq 1 \\
x_1 & \geq 30 \\
x_2 & \geq 20 \\
x_1, x_2 & \geq 0
\end{align*} \]

No feasible region exists

Some LPs are unbounded. For a max problem, an unbounded LP occurs if it is possible to find points in the feasible region with arbitrarily large z-values. This corresponds to arbitrarily large profits or revenue.

\[ \text{max } z = 2x_1 - x_2 \]
\[ \text{s.t. } \begin{align*}
x_1 - x_2 & \leq 1 \\
2x_1 + x_2 & \geq 6 \\
x_1, x_2 & \geq 0
\end{align*} \]

For a minimization problem, an LP is unbounded if there are points in the feasible region producing arbitrarily small z-values.

Consider the formulation shown to the right and the graph on the next slide.
3.2 – Graphical Solution to a 2-Variable LP

The constraints are satisfied by all points bounded by the $x_2$ axis and on or above AB and CD. The isoprofit lines for $z = 4$ and $z = 6$ are shown. Any isoprofit line drawn will intersect the feasible region because the isoprofit line is steeper than the line $x_1 - x_2 = 1$. Thus there are points in the feasible region which will produce arbitrarily large $z$-values (unbounded LP).