Chapter 2
Basic Linear Algebra

to accompany
Introduction to Mathematical Programming Operations Research,
Volume 1, 4th edition, by Wayne L. Winston and Munirpallam
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Chapter 2 - Learning Objectives

- Describe matrices and vectors with basic matrix operations.
- Describe matrices and their application to modeling systems of linear equations.
- Explain the application of the Gauss-Jordan method of solving systems of linear equations.
- Explain the concepts of linearly independent set of vectors, linearly dependent set of vectors, and rank of a matrix.
- Describe a method of computing the inverse of a matrix.
- Describe a method of computing the determinant of a matrix.
2.1 - Matrices and Vectors

Matrix – an rectangular array of numbers

\[
\begin{pmatrix}
1 & 2 \\
3 & 4
\end{pmatrix}
\quad \begin{pmatrix}
1 & 2 & 3 \\
4 & 5 & 6
\end{pmatrix}
\quad \begin{pmatrix}
1 \\
-2
\end{pmatrix}
\quad \begin{pmatrix}
2 \\
1
\end{pmatrix}
\]

Typical m x n matrix having m rows and n columns. We refer to m x n as the order of the matrix. The number in the i-th row and j-th column of A is called the i\textsuperscript{th} element of A and is written \(a_{ij}\).

\[
A = \begin{pmatrix}
a_{11} & a_{12} & \ldots & a_{1n} \\
a_{21} & a_{22} & \ldots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \ldots & a_{mn}
\end{pmatrix}
\]

Two matrices \(A = [a_{ij}]\) and \(B = [b_{ij}]\) are equal if and only if A and B are the same order and for all i and j, \(a_{ij} = b_{ij}\).

If \[
A = \begin{pmatrix}
1 & 2 \\
3 & 4
\end{pmatrix}
\quad \text{and} \quad B = \begin{pmatrix}
x & y \\
w & z
\end{pmatrix}
\]

\(A = B\) if and only if \(x = 1, y = 2, w = 3,\) and \(z = 4\)
2.1 - Matrices and Vectors

Vectors – any matrix with only one column is a **column vector**. The number of rows in a column vector is the **dimension** of the column vector. An example of a $2 \times 1$ matrix or a two-dimensional column vector is shown to the right.

$$
\begin{pmatrix}
1 \\
2
\end{pmatrix}
$$

$\mathbb{R}^m$ will denote the set all $m$-dimensional column vectors.

Any matrix with only one row (a $1 \times n$ matrix) is a **row vector**. The dimension of a row vector is the number of columns.

$$
\begin{pmatrix}
1 & 2 & 3
\end{pmatrix}
$$

2.1 - Matrices and Vectors

Vectors appear in boldface type: for instance vector $\mathbf{v}$.

Any $m$-dimensional vector (either row or column) in which all the elements equal zero is called a **zero vector** (written $\mathbf{0}$).

Examples are shown to the right.

$$
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0
\end{pmatrix}
$$
2.1 - Matrices and Vectors

Any m-dimensional vector corresponds to a directed line segment in the m-dimensional plane. For example, the two-dimensional vector \( \mathbf{u} \) corresponds to the line segment joining the point \((0,0)\) to the point \((1,2)\)

The directed line segments (vectors \( \mathbf{u}, \mathbf{v}, \mathbf{w} \)) are shown on the figure to the right.

\[
\mathbf{u} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \mathbf{v} = \begin{pmatrix} 1 \\ -3 \end{pmatrix} \quad \mathbf{w} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}
\]

The scalar product is the result of multiplying two vectors where one vector is a column vector and the other is a row vector. For the scalar product to be defined, the dimensions of both vectors must be the same.

The scalar product of \( \mathbf{u} \) and \( \mathbf{v} \) is written:

\[
\mathbf{u} \cdot \mathbf{v} = \left( u_1 \cdot v_1 \right) + \left( u_2 \cdot v_2 \right) + \ldots + \left( u_n \cdot v_n \right)
\]

Example:

\[
\mathbf{u} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \mathbf{v} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}
\]

\[
\mathbf{u} \cdot \mathbf{v} = (1 \cdot 2) + (2 \cdot 1) + (3 \cdot 2) = 10
\]
2.1 - Matrices and Vectors

Scalar multiple of a matrix: \( A = \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix} \) \( 3 \cdot A = \begin{pmatrix} 3 & 6 \\ -3 & 0 \end{pmatrix} \)

Example addition of two matrices (of same order): \( C_{i,j} = A_{i,j} + B_{i,j} \)

\[
A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -1 \end{pmatrix} \\
B = \begin{pmatrix} -1 & -2 & -3 \\ 2 & 1 & -1 \end{pmatrix}
\]

\[
C = \begin{pmatrix} 1 & 1 & 2 & -2 & 1 & -1 \\ 0 & 2 & -1 & 1 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \end{pmatrix}
\]

2.1 - Matrices and Vectors

Addition of vectors (of same degree). Vectors may be added using the parallelogram law or by using matrix addition.

\[
v = (2, 1) \\
u = (1, 2) \\
u + v = (2 + 1, 1 + 2) = (3, 3)
\]
2.1 - Matrices and Vectors

Line Segments can be defined using scalar multiplication and the addition of matrices.

If \( u = (1,2) \) and \( v = (2,1) \), the line segment joining \( u \) and \( v \) (called \( uv \)) is the set of all points corresponding to the vectors
\[
cu + (1-c)v \quad \text{where} \quad 0 < c < 1
\]

See the example to the right.

2.1 - Matrices and Vectors

Transpose of a matrix

Given any \( m \times n \) matrix

\[
A = \begin{bmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{bmatrix}
\]

\[
A^T = \begin{bmatrix}
a_{11} & a_{21} & a_{31} \\
a_{12} & a_{22} & a_{32} \\
a_{13} & a_{23} & a_{33}
\end{bmatrix}
\]

Example

\[
A = \begin{bmatrix}
1 & 2 & 3 \\
4 & 5 & 6
\end{bmatrix}
\]

\[
A^T = \begin{bmatrix}
1 & 4 \\
2 & 5 \\
3 & 6
\end{bmatrix}
\]

Observe: \( (A^T)^T = A \)
2.1 - Matrices and Vectors

Matrix multiplication

Given to matrices $A$ and $B$, the matrix product of $A$ and $B$ (written $AB$) is defined if and only if the number of columns in $A$ = the number of rows in $B$.

The matrix product $C = AB$ of $A$ and $B$ is the $m \times n$ matrix $C$ whose $ij$th element is determined as follows:

$C_{ij} =$ scalar product of row $i$ of $A \times$ column $j$ of $B$

Example

$A = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \end{pmatrix}$ $B = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$

$C_{11} = (1 \ 1) \cdot (1) = 5$ $C_{12} = (1 \ 1) \cdot (3) = 8$

$C_{21} = (2 \ 1) \cdot (1) = 7$ $C_{22} = (2 \ 1) \cdot (2) = 11$

$C = \begin{pmatrix} 5 & 8 \\ 7 & 11 \end{pmatrix}$

Note: Matrix $C$ will have the same number of rows as $A$ and the same number of columns as $B$.

---

2.1 - Matrices and Vectors

Matrix Multiplication with Excel

Use the EXCEL MMULT function to multiply the matrices:

$A = \begin{pmatrix} 1 & -1 & 2 \\ 2 & 1 & 3 \end{pmatrix}$ $B = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$

Step 1 – Enter matrix $A$ into cells $B1:D2$ and matrix $B$ into cells $B4:C6$.

Step 2 – Select the output range (B8:C9) into which the product will be computed.

Step 3 – In the upper left-hand corner (B8) of this selected output range type the formula:

$= \text{MMULT}(B1:D2,B4:C6)$.

Step 4 - Press CONTROL SHIFT ENTER (not just enter)

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<th>A</th>
<th>B</th>
<th>C</th>
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<td>8</td>
<td>$A \ B$</td>
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<td>7</td>
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</tbody>
</table>
2.2 – Matrices and Systems of Linear Equations

Consider a system of linear equations shown to the right. The variables (unknowns) are referred to as \(x_1, x_2, \ldots, x_n\) while the \(a_{ij}\)'s and \(b_j\)'s are constants. A set of such equations is called a linear system of \(m\) equations in \(n\) variables.

A solution to a linear set of \(m\) equations in \(n\) unknowns is a set of values for the unknowns that satisfies each of the system's \(m\) equations.

Example

Given \(x = \begin{pmatrix} 1 \\ 2 \end{pmatrix}\) where \(x_1 = 1\) \(x_2 = 2\)

is a solution to the system

\[
\begin{align*}
    x_1 + 2x_2 &= 5 \\
    2x_1 - x_2 &= 0
\end{align*}
\]

since (using substitution)

\[
\begin{align*}
    1 + 2 \cdot (2) &= 5 \\
    2 \cdot (1) - 2 &= 0
\end{align*}
\]

Matrices can simplify and compactly represent a system of linear equations.

\[
A = \begin{pmatrix}
    a_{11} & a_{12} & \cdots & a_{1n} \\
    a_{21} & a_{22} & \cdots & a_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix}, \quad
x = \begin{pmatrix}
    x_1 \\
    x_2 \\
    \vdots \\
    x_n
\end{pmatrix}, \quad
b = \begin{pmatrix}
    b_1 \\
    b_2 \\
    \vdots \\
    b_m
\end{pmatrix}
\]

A system of linear equations may be written \(Ax = b\) and is called its matrix representation. The matrix multiplication (using only row 1 of the \(A\) matrix for example) confirms this representation thus:

\[
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1
\]
2.2 – Matrices and Systems of Linear Equations

Ax = b can sometimes be abbreviated A|b.
For example, given:

\[
A = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad b = \begin{pmatrix} 5 \\ 0 \end{pmatrix}
\]

A|b is written:

\[
\begin{pmatrix} 1 & 2 & 5 \\ 2 & -1 & 0 \end{pmatrix}
\]

2.3 – The Gauss-Jordan Method

Using the Gauss-Jordan method, we can show that any system of linear equations must satisfy one of the following three cases:

Case 1 The system has no solution.
Case 2 The system has a unique solution.
Case 3 The system has an infinite number of solutions.

The Gauss-Jordan method is important because many of the manipulations used in this method are used when solving linear programming problems by the simplex algorithm (see Chapter 4).
2.3 – The Gauss-Jordan Method

Elementary row operations (ero).

An ero transforms a given matrix $A$ into a new matrix $A'$ via one of the following operations:

Type 1 ero – Matrix $A'$ is obtained by multiplying any row of $A$ by a nonzero scalar.

Type 2 ero – Multiply any row of $A$ (say, row $i$) by a nonzero scalar $c$. For some $j \neq i$, let:

$$\text{row } j \text{ of } A' = c \times (\text{row } i \text{ of } A) + \text{row } j \text{ of } A.$$

Type 3 ero – Interchange any two rows of $A$.

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Finding a solution using the Gauss-Jordan method.

The Gauss-Jordan method solves a linear equation system by utilizing ero’s in a systematic fashion. The steps to use the Gauss-Jordan method (with an accompanying example) are shown below:

Solve $Ax = b$ where:

$$A = \begin{pmatrix} 2 & 2 & 1 \\ 2 & -1 & 2 \\ 1 & -1 & 2 \end{pmatrix}, \quad b = \begin{pmatrix} 9 \\ 6 \\ 5 \end{pmatrix}$$

**Step 1** – Write the augmented matrix representation:

$$A|b = \begin{pmatrix} 2 & 2 & 1 & 9 \\ 2 & -1 & 2 & 6 \\ 1 & -1 & 2 & 5 \end{pmatrix}$$
2.3 – The Gauss-Jordan Method

The method uses ero’s to transform the left side of $A|b$ into an identify matrix. The solution will be shown on the right side. See $x$ to the right.

$$A|b = \begin{bmatrix} 2 & 2 & 1 \\ 2 & -1 & 2 \\ 1 & -1 & 2 \end{bmatrix}$$

Step 2 – At any stage, define a current row, current column, and current entry (the entry in the current row and column). Begin with row 1 as the current row and column 1 as the current column, and $a_{11} (a_{11} = 2)$ as the current entry.

If the current entry ($a_{11}$) is nonzero, use ero’s to transform the current column (column 1) entries to:

$$x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Steps to accomplish this were:

1. Multiply row 1 by $\frac{1}{2}$ (type 1 ero).

$$A_1|b_1 = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 2 \\ 1 & -1 & 2 \end{bmatrix}$$

2. Replace row 2 of $A_1|b_1$ by $-2^*(\text{row1} A_1|b_1) + \text{row 2 of } A_1|b_1$ (type 2 ero).

$$A_2|b_2 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & 1 \\ 1 & -1 & 2 \end{bmatrix}$$

3. Replace row 3 of $A_2|b_2$ by $-1^*(\text{row 1 of } A_2|b_2) + \text{row 3 of } A_2|b_2$ (type 2 ero).

$$A_3|b_3 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & 1 \\ 0 & -2 & 3 \end{bmatrix}$$
2.3 – The Gauss-Jordan Method

Then obtain the new current row, column, and entry by moving down one row and one column to the right and go to Step 3.

If \( a_{11} \) (current entry) would have equaled zero, then do a type 3 ero with the current row and any row that contains a nonzero entry in the current column. Use ero’s to transform column 1 as shown to the right.

If there are no nonzero numbers in the current column, obtain a new current column and entry by moving one column to the right.

Step 3 - If the new current entry is non zero, use ero’s to transform it to 1 and the rest of the column entries to 0. Repeat this step until finished.

Making \( a_{22} \) the current entry:

4. Multiply row 2 of \( A_3|b_3 \) by \(-1/3\) (type 1 ero)

5. Replace row 1 of \( A_4|b_4 \) by \(-1*(\operatorname{row} 2 \text{ of } A_4|b_4) + \operatorname{row} 1 \text{ of } A_4|b_4 \) (type 2 ero).

6. Place row 3 of \( A_5|b_5 \) by \(2*(\operatorname{row} 2 \text{ of } A_5|b_5) + \operatorname{row} 3 \text{ of } A_5|b_5 \) (type 2 ero).
2.3 – The Gauss-Jordan Method

Repeating Step 3 again for another new entry \((a_{33})\) and performing an additional three row operations yields the final augmented array:

\[
\begin{bmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 0 & 2 \\
0 & 0 & 1 & 3
\end{bmatrix}
\]

Special Cases: After application of the Gauss-Jordan method, linear systems having no solution or infinite number of solutions can be recognized.

No solution example: Infinite solutions example:

\[
\begin{bmatrix}
1 & 2 & 3 \\
0 & 0 & -2
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 0 & -1 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

2.3 – The Gauss-Jordan Method

Basic variables and solutions to linear equation systems

For any linear system, a variable that appears with a coefficient of 1 in a single equation and a coefficient of 0 in all other equations is called a basic variable.

Any variable that is not a basic variable is called a nonbasic variable.

Let \(BV\) be the set of basic variables for \(A'x=b'\) and \(NBV\) be the set of nonbasic variables for \(A'x=b'\). The character of the solutions to \(A'x=b'\) (and \(Ax=b\)) depends upon which of the following cases occur.
2.3 – The Gauss-Jordan Method

Case 1: $A'x = b'$ has at least one row of the form $[0, 0, \ldots, 0 | c]$ (c ≠ 0). Then $A'x = b'$ (and $Ax = b$) has no solution. In the matrix to the right, row 5 meets this Case 1 criteria. Variables $x_1$, $x_2$, and $x_3$ are basic while $x_4$ is a nonbasic variable.

\[
A' | b' = \begin{pmatrix}
1 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 2 & 3 \\
0 & 0 & 1 & 3 & -1 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2
\end{pmatrix}
\]

Case 2: Suppose Case 1 does not apply and NBV, the set of nonbasic variables is empty. Then $A'x = b'$ will have a unique solution. The matrix to the right has a unique solution. The set of basic variables is $x_1$, $x_2$, and $x_3$ while the NBV set is empty.

\[
A' | b' = \begin{pmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 0 & 2 \\
0 & 0 & 1 & 3
\end{pmatrix}
\]
2.3 – The Gauss-Jordan Method

Case 3: Suppose Case 1 does not apply and NBV is not empty. Then $A'x=b'$ (and $Ax=b$) will have an infinite number of solutions. BV = \{x_1, x_2, and x_3\} while NBV = \{x_4 and x_5\}.

$$A' | b' = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 3 \\ 0 & 1 & 0 & 2 & 0 & 2 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

A summary of Gauss-Jordan method is shown to the right. The end result of the Gauss-Jordan method will be one either Case 1, Case 2, or Case 3.

- Does $A' | b'$ have a row $[0, 0, ..., 0 | c]$ ($c \neq 0$)？
  - Yes
  - No
- $Ax = b$ has no solution.
- Find BV and NBV.
- Is NBV empty?
  - Yes
  - No
- $Ax = b$ has a unique solution.
- $Ax = b$ has an infinite number of solutions.
2.4 – Linear Independence and Linear Dependence

Let \( V = \{ v_1, v_2, \ldots, v_k \} \) be a set of row vectors all having the same dimension.

A linear combination of the vectors in \( V \) is any vector of the form \( c_1v_1 + c_2v_2 + \ldots + c_k \) where \( c_1, c_2, \ldots, c_k \) are arbitrary scalars.

Example:

If \( V = \{ [1, 2], [2, 1] \} \)

\[
2v_1 - v_2 = 2([1 2]) - [2 1] = [0 3]
\]

and

\[
v_1 + 3v_2 = [1 2] + 3([2 1]) = [6 3]
\]

A set of vectors is called linearly independent if the only linear combination of vectors in \( V \) that equals 0 is the trivial linear combination \( (c_1 = c_2 = \ldots = c_k = 0) \).

A set of vectors is called linearly dependent if there is a nontrivial linear combination of vectors in \( V \) that adds up to 0.

Example

Show that \( V = \{ [1, 2], [2, 4] \} \) is a linearly dependent set of vectors.

Since \( 2([1, 2]) - 1([2, 4]) = (0, 0) \), there is a nontrivial linear combination with \( c_1 = 2 \) and \( c_2 = -1 \) that yields 0. Thus \( V \) is a linear independent set of vectors.
What does it mean for a set of vectors to linearly dependent? A set of vectors is linearly dependent only if some vector in \( V \) can be written as a nontrivial linear combination of other vectors in \( V \). If a set of vectors in \( V \) are linearly dependent, the vectors in \( V \) are, in some way, NOT all “different” vectors. By "different" we mean that the direction specified by any vector in \( V \) cannot be expressed by adding together multiples of other vectors in \( V \). For example, in two dimensions, two linearly dependent vectors lie on the same line.

---

The Rank of a Matrix: Let \( A \) be any \( m \times n \) matrix, and denote the rows of \( A \) by \( r_1, r_2, \ldots, r_m \). Define \( R = \{r_1, r_2, \ldots, r_m\} \).

The rank of \( A \) is the number of vectors in the largest linearly independent subset of \( R \).

To find the rank of matrix \( A \), apply the Gauss-Jordan method to matrix \( A \). Let \( A' \) be the final result. It can be shown that the rank of \( A' \) = rank of \( A \). The rank of \( A' \) = the number of nonzero rows in \( A' \). Therefore, the rank \( A = \) rank \( A' = \) number of nonzero rows in \( A' \).
2.4 – Linear Independence and Linear Dependence

A method of determining whether a set of vectors \( V = \{ v_1, v_2, \ldots, v_m \} \) is linearly dependent is to form a matrix \( A \) whose \( i \)th row is \( v_i \). If the rank of \( A = m \), then \( V \) is a linearly independent set of vectors. If the rank \( A < m \), then \( V \) is a linearly dependent set of vectors. See the example to the right.

Given \( V = \{ [1 \ 0 \ 0], [0 \ 1 \ 0], [1 \ 1 \ 0] \} \)

Form matrix \( A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \)

After the Gauss-Jordan method:

\[
A' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]

Since the rank of \( A' = 2 \) which is \(< 3\), the set of vectors in \( V \) are linearly dependent.

2.5 – The Inverse of a Matrix

A square matrix is any matrix that has an equal number of rows and columns.

The diagonal elements of a square matrix are those elements \( a_{ij} \) such that \( i = j \).

A square matrix for which all diagonal elements are equal to 1 and all non-diagonal elements are equal to 0 is called an identity matrix. An identity matrix is written as \( I_m \). An example is shown to the right.

\[
I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\]
2.5 – The Inverse of a Matrix

For any given $m \times m$ matrix $A$, the $m \times m$ matrix $B$ is the inverse of $A$ if:

$$BA = AB = I_m$$

Some square matrices do not have inverses. If there does exist an $m \times m$ matrix $B$ that satisfies $BA = AB = I_m$, then we write:

$$B = A^{-1}$$

Example

$$A = \begin{pmatrix} 2 & 0 & -1 \\ 3 & 1 & 2 \\ -1 & 0 & 1 \end{pmatrix} \quad B = A^{-1} = \begin{pmatrix} 1 & 0 & 1 \\ -5 & 1 & 7 \\ 1 & 0 & 2 \end{pmatrix}$$

$$AB = AA^{-1} = I_3$$

To see why we are interested in the inverse of a matrix, suppose we want to solve a linear system $Ax = b$ that has $m$ equations in $m$ unknowns. Suppose that $A^{-1}$ exists. The results of multiplying both sides of the equation by $A^{-1}$ is shown to the right:

$$\left(A^{-1}A\right)x = A^{-1}b$$

$$I_m \cdot x = A^{-1}b$$

This shows that knowing $A^{-1}$ enables us to find the unique solution to a square linear system of equations.

The Gauss-Jordan method may be used to find $A^{-1}$ (or show that $A^{-1}$ does not exist).
Using the Gauss-Jordan Method to find $A^{-1}$.

Given matrix $A$, create the augmented matrix $A | I_2$.

Create the augmented matrix $A | I_2$.

Transform $A$ into $I_2$ using ero's.

Step 1 - multiply row 1 by $\frac{1}{2}$ yielding:

Step 2 - replace row 2 by $(-1)(\text{row1}) + \text{row 2}$ yielding:

Step 3 - multiply row 2 by $2$ yielding:

Step 4 - replace row 1 by $(-5/2)(\text{row1}) + \text{row 1}$ yielding:

$I_2$ has been transformed into $A^{-1}$. 

2.5 – The Inverse of a Matrix
Some matrices do not have inverses.

Consider matrix $A$ (shown to the right). Applying the Gauss-Jordan method yields $A'$.

Matrix $A$ does not have a solution since the bottom row of $A'$ has zeros. This can only happen if rank $A < 2$. If $m \times m$ matrix $A$ has rank $< m$, then $A^{-1}$ will not exist.

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$$

$$A' = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$$

Use matrix inverses to solve linear systems.

Using appropriate matrix representation to solve the system to the right.

$$A \cdot x = b \quad \text{or} \quad \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \end{pmatrix}$$

where: $A' = \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}$

$$x = A^{-1} \cdot b = \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 7 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Thus, $x_1 = 1$, $x_2 = 1$ is the unique solution to the system.
### 2.5 – The Inverse of a Matrix

Inverting Matrices with Excel

Use the EXCEL MINVERSE function to invert the matrix:

\[
A = \begin{pmatrix}
2 & 0 & -1 \\
3 & 1 & 2 \\
-1 & 0 & 1
\end{pmatrix}
\]

Enter the matrix into cells B1:D3 and select the output range (B5:D7 was chosen) where you want \( A^{-1} \) computed.

In the upper left-hand corner of the output range (cell B5), enter the formula:

\[
= \text{MINVERSE}(B1:D3)
\]

Press CONTROL SHIFT ENTER and out \( A^{-1} \) is computed in the output range.

### 2.6 – Determinants

Associated with any square matrix \( A \) is a number called the determinant of \( A \) (often abbreviated as \( \text{det} \ A \) or \( |A| \)).

For a 1 x 1 matrix:

\[
\text{det} \ A = a_{11}
\]

For a 2 x 2 matrix:

\[
A = \begin{pmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{pmatrix}
\]

\[
\text{det} \ A = a_{11}a_{22} - a_{21}a_{12}
\]

Example:

\[
\text{det} \left( \begin{pmatrix} 2 & 4 \\ 3 & 5 \end{pmatrix} \right) = 2(5) - 3(4) = -2
\]
2.6 – Determinants

If $A$ is an $m \times m$ matrix, then for any values of $i$ and $j$, the $ij$th minor of $A$ (written $A_{ij}$) is the $(m - 1) \times (m - 1)$ submatrix of $A$ obtained by deleting row $i$ and column $j$ of $A$.

\[
\begin{bmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9
\end{bmatrix}
\]

If $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ then $A_{12} = \begin{bmatrix} 4 & 6 \\ 7 & 9 \end{bmatrix}$ and $A_{32} = \begin{bmatrix} 1 & 3 \\ 4 & 6 \end{bmatrix}$.

Let $A$ be any $m \times m$ matrix. We may write $A$ as:

\[
A = \begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1m} \\
a_{21} & a_{22} & \cdots & a_{2m} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mm}
\end{bmatrix}
\]

To compute $\det A$, pick any value of $i$ ($i = 1, 2, \ldots, m$):

\[
\det A = (-1)^{i+1}a_{i1}(\det A_{i1}) + (-1)^{i+2}a_{i2}(\det A_{i2}) + \cdots + (-1)^{i+m}a_{im}(\det A_{im})
\]

The above formula is called the expansion of $\det A$ by the cofactors of row $i$. 

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2.6 – Determinants

Find det A given matrix A:

\[
A = \begin{pmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9
\end{pmatrix}
\]

det A is expanded by using row 1 cofactors (row 2 or row 3 could be used with similar results). Notice that using row 1: \(a_{11} = 1, a_{12} = 2\) and \(a_{13} = 3\) and the minors of matrix A.

\[
A_{11} = \begin{pmatrix}
5 & 6 \\
7 & 9
\end{pmatrix} \quad A_{12} = \begin{pmatrix}
4 & 6 \\
7 & 9
\end{pmatrix} \quad A_{13} = \begin{pmatrix}
4 & 5 \\
7 & 8
\end{pmatrix}
\]

Thus:

\[
det A_{11} = 5(9) - 8(6) = -3 \quad det A_{12} = 4(9) - 7(6) = -6 \quad det A_{13} = 4(8) - 7(5) = -3
\]

\[
det A = (-1)^{i+1}a_{i1}(det A_{i1}) + (-1)^{i+2}a_{i2}(det A_{i2}) + (-1)^{i+3}a_{i3}(det A_{i3})
\]

\[
= (1)(1)(-3) + (-1)(2)(-6) + (1)(3)(-3) = -3 + 12 - 9 = 0
\]

**Note:** det A = 0 makes "sense" since the set of row vectors forming matrix A are clearly linearly dependent.